Representing Interval Orders by Arbitrary Real Intervals

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Abstract

Interval orders are a way to model non-transitive indifference in comparison judgments as well as temporal relations between "events". While so far only characterizations of their representability by complete precedence vs. intersecting of *closed* or *open* (and bounded) real intervals have been known, this paper presents necessary and sufficient conditions for their representability by *arbitrary* real intervals as well as a new characterization for *open* representability. Furthermore we, like Fishburn, consider natural restrictions on representations and extend respective results of his. Accordingly, we also deal with semiorders in the sense of Luce. Interrelations of countability (separability, representability) conditions ("directly" in terms of interval orders) reveal two redundancies in Fishburn's representability conditions and indicate more direct ways to his results. The key to our results is a pair of generalizations of Fishburn's notion of "singularity".

Keywords: interval orders, semiorders, linear orders, representation theorems; preferences/comparisons, economics/mathematical psychology/measurement.

Subject Classification (MSC2000) [suggestions]**:** Primary 06A99 [ordered sets, remaining]; Secondary 06A05 [total order], 91B08 [individual preferences], 91B16 [utility theory], 91B42 [consumer behaviour, demand theory], 91C05 [measurement theory], 91C15 [one- and multidimensional scaling], 91E30 [psychophysics, ...], 91E45 [measurement & performance].

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Contents

1 Introduction and summary.

Interval orders model comparison judgments in experimental psychology (perception/evaluation of objects/persons) as well as comparison judgments concerning utilities of alternatives in economic theory—cf. [9, p. 20f.], [16, pp. 19ff.]. In both cases, indifference of judgment may well be *non-transitive* (in these realms, however, interval orders like to be replaced by the somewhat more special "semiorders", cf. [13, 16]). "Utility functions" are (in general) not applicable in such situations. Interval orders also have served to model temporal relationships among events in philosophical ontology (see references in [19, 20]) and in computer science. (For further applications see [references in] [16, pp. 19ff.], [9, pp. 19ff.], or [17, p. 255].)[1] They may be defined to be representable by the relationships of complete precedence and of intersecting between intervals of linear orders.

Which interval orders may, in particular, in this manner be represented by intervals with respect to the linear 'less than'-relation between *real numbers*? This question seems to be held important for the applicability of mathematics to the sciences (at least to the ones mentioned above), and, actually, has occupied all non-philosophical published work on "representability" of interval orders. Such a representation may be considered an appropriate variant of a utility function and would, in an extended manner, fit the research programme of "classical" measurement theory (cf. [11]), where everything "empirical" must be tried to be viewed as something made out of the real numbers.[2]

While the literature so far only has characterized interval orders representable by complete precedence vs. intersecting of *closed* or *open* (and

[1] As further examples archeology and paleontology are listed, where the duration of such "events" may exceed lifetimes.

[2] Reservations against this research programme—as espoused, e.g., in [14] —are not considered an obstacle to happily presenting here results fitting it.

bounded) real intervals (usually equivalently considering *pairs of functions* and involving questions of [semi-]continuity of interval borders with respect to some topology on the alternatives), this paper presents necessary and sufficient conditions for their representability by *arbitrary* real intervals (disregarding continuity, however). Moreover, a new characterization of representability by open real intervals is offered.

Some of the countability and "separability" conditions involved in these characterizations of real representability interrelate. These interrelations yield alternative or even shorter proofs of some of Fishburn's [8, 9, Sec.s 7.5f.] results and add alternative ("logically/practically weaker") solutions to a characterization problem previously solved by Fishburn and [15].

We shall begin with briefly reporting these results on "closed" real representability known so far that do not consider continuity of interval borders, only preceded by a minimal amount of conventions required in that report, and followed by some improvement concerning real representability of semiorders. We shall then present our characterizations of "arbitrary" and of "open" real representability (starting with a brief report of Fishburn's [8, 9] contribution and closing with an improvement of his characterization result). The "position" of one of Fishburn's representability conditions, and two aspects of "minimality" of representations will be discussed briefly. We shall finish presentation of own results by considering extensions of theorems of Fishburn's [8, 9] concerning semiorders, taking into account those "minimality" notions. We shall briefly wonder about empirical significance of our results. After extending notational machinery and order-theoretical background, we shall proceed by the obvious explanations concerning "unbounded" real representations and "function pair representations", then by proving the remaining claims, concerning interrelations between representability (separability, countability) conditions first, concerning necessity of characterizing conditions next, and concerning sufficiency finally.

2 First preliminaries.

2.1 Binary relations.

If $R \subseteq X \times X$ (R a binary relation on the largest set X mentioned so far), we write $x\ R\ y$ for $(x, y) \in R$, by $\sim_R$ we denote $X \times X \setminus (R \cup R^{-1})$ (the *symmetric complement* of R),[3] and $\precsim_R$ is $R \cup \sim_R$. We shall use the latter notation for *asymmetric*[4] relations R only. For these, propositional logic obtains that $a \precsim_R b$ just means *not* $b\ R\ a$ ($\precsim_R$ is the "reverse negation" of R).

[3] By R^{-1} we mean the binary relation defined by $x\ R^{-1}\ y$ *iff* $y\ R\ x$.—Some notations are vague concerning X, but this will not matter.

[4] R being asymmetric means $R \cap R^{-1} = \emptyset$.

A set Y will be called *R-separable* if there is some countable subset $D \subseteq Y$ such that for all $x, y \in Y$ such that $x\ R\ y$ there is $d \in D$ such that $x \precsim_R d \precsim_R y$.—We shall apply this notion to a class of (in particular: asymmetric) relations only, for which two other definitions of separability notions given in [3, 1.4.3.f.] and used in other work on real representability of binary relations are equivalent.[5] In this respect, the notion is somewhat "more widely applicable than it seems at first".

If a statement is logically or set-theoretically equivalent to a statement only involving X and R, replacing R by its converse R^{-1} yields the *dual* statement. If notions *derived* from X and R are involved, it will be clear what the *dual* is, as well. It will also (at any instance) be clear what the dual of a *notion* is. We shall often deal with pairs of assumptions or notions dual to each other. In such cases, reasonings can be dualized as well, and we shall treat only one version explicitly. Cf. [1, p. 13].

We write RR' for the composition of any binary relations R, R'.[6]

2.2 Interval representations and interval orders.

A binary relation L (strictly) *linearly orders* some set Q—and (Q, L) will then be called a *linear order*—, if $L \cap (Q \times Q)$ is irreflexive, transitive and trichotomic (connex, "complete").[7] A (corresponding) *interval* (a non-void "convex" subset) is a non-void subset H of Q such that for all $x, y \in H$ each situation $x\ L\ z\ L\ y$ implies $z \in H$. $\mathcal{I}(Q, L)$ will denote the set of these intervals.

If R is some binary relation and X some set, some ρ will be said to be an *interval representation* (*IR*) of (X, R) in (Q, L) if L linearly orders Q and if ρ is a map $X \to \mathcal{I}(Q, L)$ such that

$$x\ R\ y \qquad \text{iff} \qquad \forall p \in \rho(x) \quad \forall q \in \rho(y): \quad p\ L\ q.$$

$\rho[A]$ (the *image* of A under ρ) may contain only *closed* and *bounded* intervals (i.e., they contain their boundary points); ρ will then be called a *closed and bounded* IR (*CBIR*) itself.

By [9, Thm. 2.6], an interval order may now be defined to be an ordered pair (X, R) which has a CBIR in some linear order. In fact, from the result cited one can derive that any (X, R) having an IR has a CBIR. Therefore, an interval order can as well be defined by the property of just having some IR.

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[5] In fact, while Fishburn's [8] uses the definition given above, his [9] uses one of these two other definitions. More precisely: he uses the respective *notions*—never actually using the term 'separable'. The difference stems from different definitions of '(order-)dense'—cf. our Subsection 4.8 below. As a consequence, the appearence that [8] and [9, Sec.s 7.4–7.6] would state and prove the same theorems would not be quite correct. However, we shall ignore the difference in the sequel.

[6] I.e., for $\{ (x, y) \mid \exists z : x\ R\ z\ R'\ y \}$.

[7] To be explicit, trichotomicity of R means that *whenever* $x \neq y$, $x\ R\ y$ or $y\ R\ x$.

As a simple example, the intervals of a linear order (Q, L) together with their relation of "complete precedence" with respect to L[8] trivially form an interval order—the *natural* interval order of (Q, L). (The term 'natural interval order' will also be used for suborders of such interval orders.) An IR of (X, R) may then just be viewed as a "strong"[9] homomorphism from (X, R) to the natural interval order of some linear order, and "general" interval orders are just those (X, R) from which there is such a strong homomorphism into some *natural* interval order.

From now on, we deal especially with one arbitrary interval order (A, T) *where* $T \subseteq A \times A$. A may well be infinite (otherwise representability problems to be dealt with would be trivial); but A must not be void. Introducing I as $\sim_T$ gives rise to the well-known composite binary relations IT and TI on A, properties of which will be recalled below.

2.3 Visualization.

However, at this stage some visualization of IT and TI may be helpful. Assume *for a moment* A is a three-element set $\{a, b, c\}$ and $T = \{(a, b)\}$. This situation may be visualized by Figure 1. $a\ T\ b$ is visualized by arranging

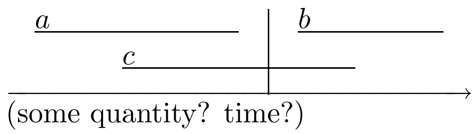

Figure 1: $A = \{a, b, c\}$, $T = \{(a, b)\}$.

horizontal strokes representing a and b so that a *vertical* stroke can be filled in right-hand to the horizontal stroke representing a and left-hand to the horizontal stroke which represents b.

In contrast, no vertical stroke would have two horizontal strokes on different sides such that one of them would represent a and the other would represent c. This holds for b in place of a, as well. Rather, c is "overlapping" a as well as b—this is $a\ I\ c\ I\ b$.

Furthermore, that c "starts at the left of" b is "witnessed" by a overlapping c but wholly preceding b—this is $c\ IT\ b$. Similarly, a "*ends* to the left of where c ends", "witnessed" by b overlapping c but being wholly preceded by a—this is $a\ TI\ c$.[10]

However, one *cannot* "read" $a\ IT\ c$ from the diagram, since there is no "witness" d such that $a\ I\ d\ T\ c$.

[8] To be precise, this relation $!L!$, say, of "complete precedence" with respect to L is meant to be defined by, for $H_0, H_1 \in \mathcal{I}(Q, L)$, $H_0\ !L!\ H_1$ iff, for all $p \in H_0$ and all $q \in H_1$, $p\ L\ q$.

[9] Cf. [3, 1.1.9].

[10] In the interpretation of the situation as temporally related events, c *begins* earlier than b, while a *ends* earlier than c.

With the latter reservation, such visualizations show all properties of T, I, IT, and TI that will be needed, and they may be helpful in checking and understanding respective claims below.

3 Known and new results.

3.1 "Closed" representability as known.

Particular interest has been enjoyed by those CBIRs of (A, T) in $(\mathbf{R}, <)$, the "less-than" ordering of the real numbers—we call them *closed bounded real interval* representations (*CBRIR*s). A *real interval* is just meant to be an element of $\mathcal{I}(\mathbf{R}, <)$.

CBRIRs usually appear as *function pair* representations (*FPR*s), i.e., as maps $u, v : A \to \mathbf{R}$ such that

$$a\ T\ b \quad \text{iff} \quad v(a) < u(b).$$

($u \leq v$ follows.) More precisely, if F is a CBRIR, $u : A \to \mathbf{R}, a \mapsto \inf F(a)$ and $v : A \to \mathbf{R}, a \mapsto \sup F(a)$ form a FPR; and if (u, v) is a FPR, $a \mapsto [u(a), v(a)]$ is a CBRIR.[11] So existence of a FPR is the same as existence of a CBRIR. Mathematical economy typically asks for such representations by (semi-)*continuous* functions with respect to some topology on A (which may be viewed, e.g., as a subset of $\mathbf{R}^n$)—see [3, Chap. 6], in particular for Chateauneuf's solution of a respectively restricted version of the problem that the present subsection deals with.

Fishburn ([8, 9, Sec. 7.5]) presented a necessary and sufficient condition for the existence of a CBRIR that can be stated as a conjunction of two parts. The first of them is:

(F1) *A be IT- as well as TI-separable.*

Unlike pairs of countability conditions dual to each other the equivalence of which turns out below, IT- and TI-separability are independent.[12] There *is* some redundancy in the above presentation of (F1), however, which will be revealed in Subsection 6.5 below.

Knowing that $\sim_{TI}$ is an equivalence relation and that a *singular* element a of A is (by Fishburn's [8] definition)[13] one for which $b\ I\ a\ I\ c$ implies $b\ I\ c$, the second part of Fishburn's condition can be stated thus:

[11] According to widespread use, $[u(a), v(a)]$ here denotes $\{ r \in \mathbf{R} \mid u(a) \leq r \leq v(a) \}$, of course.

[12] An example for each direction of non-implication is provided by the "self-representing" interval order of all (bounded) real intervals which contain their lower [upper, resp.] boundary. Thus, $[r, s)$ as well as $[r, s]$ occur in one of the two examples. The example depends essentially on the fact that, e.g., $[s, s]$ "witnesses" the former two intervals ending differently upwards.

[13] In [9], Fishburn uses the term 'simplicial' instead for some graph-theoretical reason.

(F2) *There be only countably many TI-equivalence classes*[14] *which contain no singular element and for an element b of which the set of a such that $b\ T\ a$ has some IT-minimum, but no singular one.*[15]

One may feel uneasy with such counting equivalence classes in a characterization of some interval orders. To avoid it, one may (at the cost of assuming the Axiom of Choice)[16] word (F2) equivalently thus: *Every subset C of A which is linearly ordered by TI, which contains no singular element, and for every element c of which the set of a such that $c\ T\ a$ has some IT-minimum, but no singular one, be countable.*

Oloriz, Candeal and Induráin [15] from the (Public) Universities of Pamplona and Zaragoza (Spain), obviously not aware of Fishburn's result, more recently presented *another* necessary and sufficient existence condition, viz.

(E) *there be a countable $D \subseteq A$ such that for all $(a, b) \in T$ there is some $d \in D$ such that $a\ T\ d \precsim_{IT} b$.*[17]

As a "test" that both Fishburn and the Spanish authors are right (if one of them is), one may reason just in terms of (A, T) that the conjunction of Fishburn's conditions (F1), (F2) is equivalent to the "Spanish" condition (E). This will be done (not as a "test", but just to show it) in Subsection 6.3 below.

A slightly weaker characterization has been presented by Doignon *et al.* [6, Prop. 9f.] already in 1984, using

(D1) *there be a countable $D \subseteq A$ such that for all $(a, b) \in T$ there is some $d \in D$ such that $a\ T\ d \precsim_{IT} b$ or $a \precsim_{TI} d\ T\ b$.*

One could drop the restriction of boundedness and consider *CRIR*s—note that, in topological terminology, a real interval having just one boundary which it contains counts as *closed* (as a closed *set*, at least). By the same token, even $\mathbf{R}$ is a "closed" real interval.—Clearly, CRIRs do not really make a difference; though the following summary including them might be welcome.[18]

[14] TI-equivalence, of course, is meant to be $\sim_{TI}$—since this is known to be an equivalence relation (as will be indicated below or clear from the "visualization" above).

[15] Fishburn's way of stating this condition may be more perspicuous, but the way we have presented it better fits into the framework of the present paper. Some notation and reasoning introduced below should, however, improve perspicuity of (F2)—see Subsection 6.2.

[16] Cf., e.g., [12]. The present author is, up to now, not able to see how the *Countable* Axiom of Choice could suffice for *both* directions to be proved.

[17] At least, this is a way of stating the condition the present author prefers. We shall return to this "translation problem" in Subsection 6.3.—IT is, in fact, asymmetric, so $d \precsim_{IT} b$ means *not* $b\ IT\ d$ as well as $d\ (IT \cup \sim_{IT})\ b$ here.

[18] Literally, Fishburn's [8] deals with 'closed-interval representability' in the sense of existence of a CRIR: In proving his Thm. 5 on [8, p. 102], he takes care of *unbounded* in-

Theorem 1 (Fishburn; Oloriz / Candeal / Induráin). *The following conditions are equivalent.*

(i) *(A, T) has a CBRIR;*

(ii) *(A, T) has a CRIR;*

(iii) *(A, T) has a FPR;*

(iv) *(F1) and (F2) hold;*

(v) *(E) holds;*

(vi) *(D1) holds.*

In this summary, however, Conditions (F1) and (F2) may be replaced by somewhat weaker ones. This will be done in Theorem 7 in Subsection 3.3 and in Theorem 11 in Subsection 6.5.

Following Fishburn [8], we could have told more in the following way: In place of assuming (A, T) to be an interval order, we could have assumed nothing more than $T \subseteq A \times A$—at the same time augmenting conditions (iv) through (vi) by *and (A, T) is an interval order.* Indeed, each of the first two conditions of Theorem 1 (under the parsimonious assumption) implies that (A, T) is an interval order according to Subsection 2.2.—This remark applies respectively to our representation theorems of subsections 3.3 and 3.5–3.7 below.[19]

3.2 Some improvement concerning semiorders.

Following Luce [13], (A, T) is a *semiorder* if (additionally to being an interval order) $a\ T\ c\ T\ b$ excludes $a\ I\ d\ I\ b$, i.e., no representing interval intersects each of *three* pairwise non-intersecting representing intervals. This may

tervals. On p. 92 of the same paper, he defines "closed-interval representable" by existence of a RIR (cf. Subsection 3.3) such that taking the closures of the image intervals yields a RIR again (assuming 'inf $A = \infty$' in case A has no lower bound etc.). By his own understanding, such intervals just need not be bounded! On p. 95, however, he assumes without explanation that, for a given 'closed-interval representation' ρ, there is pair of maps u, v such that $\rho(x) = [u(x), v(x)]$.—In [9], in contrast, Fishburn defines "closed-interval representable" explicitly by existence of a FPR (in the sequel telling the same story as in [8]).—These observations indicate that Fishburn (already in [8]) considers closed intervals bounded (this may be common use), while he may have overlooked something on his [8, p. 92] (e.g., in writing the sentence following his (2), he may have thought that "closing open ends" turned RIRs into CBRIRs, disregarding unbounded image intervals). However, there is no plain error; it only might have been appropriate to supplement the reasoning of p. 95 by a similar hint on monotone transformations with bounded image set as on p. 102 in order to explain that his 'closed-interval representable' entails existence of a CBRIR.

[19] We cling to displaying the weaker versions for seeming better readability of the theorems and streamelinedness of the respective proofs.

be understood as representing intervals having "similar length", maybe in connection with a constant threshold of noticeable differences with respect to some quality of experimental stimuli or to subjective utility.

For semiorders, the relation

$$T_0 := IT \cup TI$$

is of particular importance. In Fishburn's [8, 9], representability theorems for interval orders involving (F1) are followed by respective variants for semiorders where (F1) is replaced by T_0-separability. In proofs of such variant theorems, Fishburn states that T_0-separability (obviously) implies (F1), whereas he proves necessity of T_0-separability for existence of a CBRIR just starting at the assumption of such a CBRIR (or FPR). Instead, he simply could have used (proved)

Theorem 2. *If (A, T) is a semiorder, A is T_0-separable iff (F1) holds.*

3.3 Results on other kinds of representations.

Besides C(B)RIRs, Fishburn ([8, 9, Sec. 7.5]) considered *representations by arbitrary* (bounded) real intervals (*arbitrary (bounded) real* IRs, *ARIR*s/ *ABRIR*s). In some contexts, we shall write *RIR* for *real interval representation* in place of ARIR.

Fishburn only found that his (F1) is a *sufficient* condition for their existence. That it is not *necessary* becomes obvious by taking (A, T) to be the "self-representing" natural interval order made up by the set of all (bounded) real intervals, where (F1) is *not* satisfied.[20] Note that a set $D \subseteq A$ according to the definition of IT-separability (which is required by (F1)) must contain an element of at least one of two "neighbouring" IT-equivalence classes. By contrast, in the previous example each real serves as a boundary contained in intervals of one IT-equivalence class as well as a boundary *not* contained by the elements of *another* IT-equivalence class. So there are *uncountably* many "jumps" of IT-equivalence classes, and a D according to the definition of IT-separability cannot exist. Dually, of course, neither is the A specified above TI-separable.

However, if (in the same example still) we restrict consideraration to real intervals (i) containing or (ii) *not* containing their lower boundaries, IT-separability obtains ("separately") in both cases (i) and (ii). Dually, TI-separability obtains in both of the analogous two cases concerning upper boundaries. By additionally generalizing Fishburn's notion of *singular* elements, this idea turns out to be the key to solving the characterization problem for A(B)RIRs.

[20] Fishburn [8] is aware that (F1) is not necessary and presents another example than is presented here.

The proof strategy developing in this way, furthermore, turns out to solve another problem to be posed now. Namely, instead of representations by *closed* real intervals one might prefer representations by *open* real intervals. Again, one may additionally consider the restriction of boundedness, and it is clear what *ORIR*s and *OBRIR*s are meant to be now. Just in order to complete the picture, one may additionally consider *weak function pair* representations (*WFPR*s), defined as pairs (u, v) of maps $u, v : A \to \mathbf{R}$ such that

$$a\, T\, b \quad \text{iff} \quad v(a) \leq u(b)$$

(now $u < v$ follows)—although it is clear from the start what is to be said about them.

Doignon *et al.* [6, Prop. 8] already considered the problem of which interval orders have O(B)RIRs. They characterized such interval orders by the condition

(D2) *there be a countable* $D \subseteq A$ *such that for all* $(a, b) \in \precsim_T$ *there is some* $d \in D$ *such that* $a \precsim_T d \precsim_{TI} b$ *or* $a \precsim_{IT} d \precsim_T b$.

We find another solution of the same problem in quite different terms.[21]

The generalizations announced above are the following.[22] Call an element a of A *lower-singular* if $\{\, b \mid a\ I\ b \,\}$ has a TI-minimum. (If a is *singular* in Fishburn's sense, a is such a TI-minimum itself.) Dually, call a *upper-singular* if the same set has an IT-maximum. (Again, if a is *singular* in Fishburn's sense, a is such an IT-maximum itself.)

Thus, singular elements are both lower- and upper-singular; the converse does not hold: the element c in the above diagram is a counter-example. If (A, T) is finite, all $a \in A$ are both lower- and upper-singular. (Subsection 3.6 provides a further elucidation still before diving into proofs.)

Let S_- be the set of *lower-singular*, S_+ be the set of *upper-singular* elements of A. We shall find:

Theorem 3. *S_- is IT-separable iff S_+ is TI-separable.*

In this case we shall call (A, T) *singular-separable*.

Theorem 4. *There are only countably many IT-equivalence classes containing lower-singular elements iff there are only countably many TI-equivalence classes containing upper-singular elements.*

[21] Indeed, the author considered and solved the problem without knowing of [6]. He was lead to the problem by Peter Schuster's (member of the Mathematical Institute at Munich University) suggestion.

[22] There is a precursor to these generalizations in Fishburn's proof of his characterization of interval orders having CBRIRs—cf. Fn. 35 and Proposition 1 below. Moreover, they appear in disguise in his proof that (F1) implies existence of an ABRIR.

In this case we shall call (A, T) *singular-countable*. Again, mentioning equivalence classes may be avoided by stating the condition equivalently as *every subset of* S_- *linearly ordered by* IT *be countable* or as *every subset of* S_+ *linearly ordered by* TI *be countable*.

We call (A, T) *regular-separable* if $A \setminus S_-$ is IT-separable and $A \setminus S_+$ is TI-separable.[23]

The terms thus explained allow the following solutions of the characterization problems introduced before. Theorem 5 is meant to be the main result of the paper.

Theorem 5. *The following conditions are equivalent.*[24]

(i) (A, T) *has an ABRIR;*

(ii) (A, T) *has an ARIR;*

(iii) (A, T) *is singular- and regular-separable.*

Theorem 6. *The following conditions are equivalent.*[25]

(i) (A, T) *has an OBRIR;*

(ii) (A, T) *has an ORIR;*

(iii) (A, T) *has a WFPR;*

(iv) (A, T) *is singular-countable and regular-separable;*

(v) *(D2) holds.*

Finally, as announced at the end of Subsection 3.1, the new singularity notions allow confining the countability condition (F2) occuring in Condition (iv) of Theorem 1 to an (in general proper) *subset* of the set of equivalence classes that (F2) is about in the following way.

Theorem 7. *Each of the conditions of Theorem 1 is equivalent to the conjunction of (F1) with*

(G) *there be only countably many* TI*-equivalence classes which contain no* upper-*singular element and for whose elements* a *the set of* b *such that* $a\ T\ b$ *has some* IT*-minimum, but no* lower-*singular one.*[26]

[23] The intervals of the lexicographic order on $\mathbf{R} \times \mathbf{N}$ refute the conjecture that the two defining conditions might be equivalent by analogy to the case of singular-separability.

[24] To elucidate the second remark after Theorem 1, the equivalence even holds with assuming that (A, T) be an interval order only in (iii).

[25] Again, the equivalence holds with assuming that (A, T) be an interval order only in (iv) and (v).

[26] Since, as noted above, a singular element is both lower- and upper-singular, a *set* containing *no* lower- or no upper-singular element *is* a set containing no singular element—therefore the set of equivalence classes mentioned here is a subset of that mentioned in (F2), indeed.

By analogy to Subsection 3.1, (G) could equivalently be worded thus: *Every subset of A which is linearly ordered by TI, which contains no* upper-*singular element, and for every element c of which the set of a such that $c\ T\ a$ has some IT-minimum, but no* lower-*singular one, be countable.*

Theorem 7 is meant to show that our singularity notions are somewhat more fundamental than Fishburn's singularity notion: Our notions tie (F2) to a(n in general) smaller set; more generally, they help to solve the same problems as Fishburn's notion as well as new problems.

3.4 Conditions (F1) etc.

The "relative position" of (F1) with respect to Condition (*iii*) of Theorem 5 and to Condition (*iv*) of Theorem 6 is the following.

Theorem 8. *Of the following conditions, each implies the next one without itself being implied by it.*

(*i*) (A, T) *is singular-countable and regular-separable;*

(*ii*) *(F1) holds;*

(*iii*) (A, T) *is singular- and regular-separable.*

By Theorem 5, this—or rather our proof of it—yields another insight into Fishburn's result that (F1) is sufficient for the existence of an A(B)RIR. Another immediate consequence is

Corollary 1. *Each of the conditions of Theorem 6 implies each of the conditions of Theorem 5—but not conversely.*[27]

3.5 Real homomorphisms, "exact" representations.

This subsection provides at least one definition of vital importance and prepares some generalizations of some of Fishburn's [8, 9] results. It is concerned with representing IT and TI in some strict manner by RIRs.

We say that φ is a (strong) *real R-homomorphism on X* if it maps X into $\mathbf{R}$ and R is some binary relation such that for all $x, y \in X$

$$x\ R\ y \quad \text{iff} \quad \varphi(x) < \varphi(y).$$

For the sequel one might particularly keep in mind that a real R-homomorphism on X maps $x, y \in X$ to the same real iff $x \sim_R y$.

Furthermore, if (Q, L) is a linear order and $H, H' \in \mathcal{I}(Q, L)$, then H *(L-)exceeds H' below* if $p\ L\ q$ for some $p \in H$ and all $q \in H'$. Call an IR

[27] The implications, of course, follow directly from the fact that singular-countability implies singular-separability or, therefore, that Condition (*iv*) of Theorem 6 implies Condition (*iii*) of Theorem 5. Only that the converses do not hold needs Theorem 8.

ρ of (A, T) in (Q, L) *lower-exact* if, for $a, b \in A$, $\rho(a)$ exceeding $\rho(b)$ below implies $a\ IT\ b$.[28]

Dually, H *(L-)exceeds* H' *above* if $q\ L\ p$ for some $p \in H$ and all $q \in H'$. Call an IR ρ of (A, T) in (Q, L) *upper-exact* if, for $a, b \in A$, $\rho(a)$ exceeding $\rho(b)$ above implies $b\ TI\ a$.[29]

Call ρ *exact* if it is both lower- and upper-exact.

An exact IR of (A, T) maps $a, b \in A$ to the same interval iff a, b are both IT- and TI-equivalent[30] (whereas this equivalence does not suffice for exactness). As another elucidation, if an IR of (A, T) in (Q, L) is exact, it is a "strong" homomorphism of (A, IT) and of (A, TI) to the respective structures derived from the natural interval order of (Q, L).[31] The converse does not hold.

The proof of [9, Thm. 2.6] shows that *every* interval order has an *exact* IR. Readers acquaintant with the matter will know a number of IRs without ever having thought of a non-exact one. A CRIR or ORIR ρ is lower-exact iff $\inf \rho(a) = \inf \rho(b)$ whenever $a \sim_{IT} b$, etc. Concerning existence of exact CRIRs, Fishburn [8, 9] told everything. We extend his theorem in Subsection 3.6 and thus postpone the matter. Fishburn did not deal with exactness in the case of ARIRs. *We* shall do in the same Subsection 3.6. At this place, we just mention what concerns ORIRs:

Corollary 2. *The following conditions are equivalent.*[32]

- (i) (A, T) *has an ORIR;*
- (ii) (A, T) *has an* exact *OBRIR;*
- (iii) (A, T) *has a WFPR* (u, v) *such that* u *is a real* IT*-homomorphism on* A *and* v *is a real* TI*-homomorphism on* A.

3.6 Representing singularity.

Fishburn ([8, Thm.s 2, 5] [9, Thm.s 7.6, 7.10]) found that, if a CBRIR exists,[33] then there is an exact one which maps each singular element of A to a singleton; and if an ABRIR exists, then there is one which maps each singular element of A to a singleton (nothing said about exactness).

[28] By Lemma 2 below, the converse implication holds anyway.

[29] Again by Lemma 2 below, the converse implication holds anyway.

[30] This may be considered an aspect of "minimality" according to Section 1: If ρ is exact and, e.g., $a \sim_{IT}$, then $\rho(b)$ does not need "additional" points below $\rho(a)$.

[31] To be precise, let $!L!$ be as defined in Fn. 8. An exact IR of (A, T) in (Q, L), then, is a strong homomorphism from (A, IT) to $(\mathcal{I}(Q, L), \sim_{!L!} !L!)$ and from (A, TI) to $(\mathcal{I}(Q, L), !L! \sim_{!L!})$.

[32] According to the second remark after Theorem 1, the equivalence holds without assuming that (A, T) be an interval order.

[33] For exegetical doubts cf. Fn. 18, for their irrelevance cf. Theorem 1.

These results can be extended to lower- and upper-singularity, to ORIRs and concerning exactness as follows.

It is easy to see that, for $a, b \in A$, b is an IT-maximum of $\{ c \mid a \; I \; c \}$ iff a is a TI-minimum of $\{ c \mid b \; I \; c \}$.[34] Define $a \; I^* \; b$ to hold then.[35] a then is a "witness" for b being lower-singular, and, dually, b is a "witness" for a being upper-singular.[36] (Now a is singular in Fishburn's [8] sense iff $a \; I^* \; a$.)[37]

Call an IR ρ of (A, T) in some linear order (Q, L) *singular-exact* if, (exactly)[38] for $(a, b) \in I^*$, $\rho(a)$ and $\rho(b)$ have exactly one element in common.[39]

Our announced extensions then are:

Corollary 3. *The following conditions are equivalent.*[40]

(*i*) (A, T) *has a CRIR;*

(*ii*) (A, T) *has an* exact and singular-exact *CBRIR;*

(*iii*) (A, T) *has a FPR* (u, v) *such that* u *is a real* IT*-homomorphism on* A*,* v *is a real* TI*-homomorphism on* A*, and* $u(b) = v(a)$ *iff* $a \; I^* \; b$*.*

We could have dealt with exactness and singular-exactness separately to tell more (cf. Corollary 2 above). This is done in Section 5.

Corollary 4. (A, T) *has an ARIR iff it has an* exact and singular-exact *ABRIR.*[41]

In contrast, exact and singular-exact O(B)RIRs exist trivially, if at all:

Corollary 5. *If* (A, T) *has an ORIR, the following conditions are equivalent.*[42]

(*i*) (A, T) *has a(n exact and) singular-exact O(B)RIR;*[43]

(*ii*) $S_- = \emptyset$*;*

(*iii*) $S_+ = \emptyset$*.*

---

[34] Cf. Proposition 1 in Subsection 4.5.

[35] $I \setminus (I^* \cup (I^*)^{-1})$ is what Fishburn calls I in [8, (10)] and [9, p. 137].

[36] See Proposition 1 again.

[37] See Lemma 5.

[38] If ρ is *any* IR, $\rho(a) \cap \rho(b)$ is *no* singleton if *not* $a \; I^* \; b$. This follows from Lemma 2 below.

[39] This quite obviously deserves to be called a "minimality" condition.

[40] Again, the equivalence holds without assuming that (A, T) be an interval order.

[41] Again, the equivalence holds without assuming that (A, T) be an interval order.

[42] The second remark after Theorem 1 makes no difference here, since existence of an ORIR implies that (A, T) is an interval order, anyway.

[43] There is no interesting WFPR version analogous to the FPR version in Corollary 3.

The parentheses are meant to indicate that singular-exactness is the problem, not exactness or boundedness.

In fact, Fishburn's and our proofs of representability claims show that (A, T) has some exact and singular-exact CIR as well as some exact and singular-exact AIR in *any* case (it will be clear what these abbreviations and the following one are to be meant for "abstract" IRs); and that it has some singular exact OIR iff $S_- = \emptyset$ or, equivalently, $S_+ = \emptyset$.[44]

3.7 Extending Fishburn's results on semiorders.

In [8, p. 97], Fishburn defines a RIR ρ (of (A, T)) to be *monotonic* if $\inf \rho(a) < \inf \rho(b)$ and $\sup \rho(a) > \sup \rho(b)$ for *no* $a, b (\in A)$. He then states two theorems on existence of such RIRs for semiorders.[45] For ARIRs, however, there seems to be a more adequate notion: Call an IR ρ of (A, T) in some linear order (Q, L) *semiorderlike* if $\rho(a)$ L-exceeds $\rho(b)$ below *and* above for *no* $a, b \in A$. Now, monotonicity and semiorderlikeness are the same for CRIRs as well as for ORIRs, whereas for ARIRs semiorderlikeness only *implies* monotonicity—the converse not holding.

Rather obviously:

Corollary 6. *Any exact RIR of (A, T) is semiorderlike (and, hence, monotonic) iff (A, T) is a semiorder.*[46]

Since in all his proofs of theorems mentioned so far, Fishburn presents *exact* RIRs, Corollary 6 (or its proof) is suggested to be an alternative to Fishburn's proofs of his theorems on monotonicity.

Moreover, Fishburn's results might be extended in various ways according to the two previous subsections, e.g.:

Corollary 7. *In corollaries 2 through 5, 'exact' may be replaced by 'exact and semiorderlike' iff (A, T) is a semiorder.*

To summarize: Closed/open/arbitrary representability implies *exact* representability of the respective kind; and in case of semiorders, it implies respective *semiorderlike* (*a fortiori:* monotonic) representability. The slogans are: Exactness is no problem; and for semiorders, semiorderlikeness (monotonicity) is no problem.

Furthermore, Fishburn [8, p. 97] defines a RIR ρ to be *strictly monotonic* if, for all $a, b \in A$, $\inf \rho(a) < \inf \rho(b)$ iff $\sup \rho(a) < \sup \rho(b)$. He states that

[44] In the present paper, we omit a third kind of "exactness" ("minimality"): An IR ρ might be required to meet that, if $a\ T^*\ b$, then $\rho(a), \rho(b)$ should be "as tight as possible" (unless $a \notin S_+$, $b \notin S_-$; $\rho(a) \cap \rho(b)$ interval [but for one point]). Our proof of existence of an ARIR, however, shows existence of an ARIR meeting all three kinds of "exactness".

[45] His [9, Sec.s 7.5f.] essentially states the same theorems without explicitly introducing the notion of monotonic RIRs.

[46] Again, the equivalence holds without assuming that (A, T) be an interval order.

a strictly monotonic CRIR exists where any CRIR exists, but confesses to have been unable to decide the case of ARIRs. Indeed, for ARIRs the question may be somewhat ill-posed, since, again, a perhaps more adequate notion presents itself: Call an IR ρ of (A, T) in some linear order (Q, L) *strictly semiorderlike* if, for any $a, b \in A$, $\rho(a)$ L-exceeds $\rho(b)$ below iff $\rho(b)$ L-exceeds $\rho(a)$ above. Now like above, strict monotonicity and strict semiorderlikeness are the same for CRIRs as well as for ORIRs, whereas for ARIRs strict semiorderlikeness only *implies* monotonicity—the converse not holding.[47]

In the present paper at least, we refuse to extend Fishburn's theorem concerning this notion.[48] Besides a space-time reason, there is a moral reason:

Corollary 8. *Assume (A, T) is a semiorder having an exact RIR ρ. Then ρ is strictly semiorderlike iff $IT = TI$.*

(This does *not* hold for strict monotonicity in place of strict semiorderlikeness.) We deem exactness a very high virtue; therefore, we consider a strictly monotonic RIR ρ of (A, T) "cheating" unless $IT = TI$. On the other hand, if $IT = TI$, (since exactness is granted by earlier results) strict monotonicity deserves no particular attention. Furthermore, the method of creating IRs used in the present paper *only* yields *exact* IRs (cf. Corollary 8) and is not as apt to create "extra points" needed for strict semiorderlikeness as Fishburn's is.[49]

Nevertheless, in contrast to the case of strict monotonicity, we are able to decide whether existence of a RIR entails existence of a strictly *semiorderlike* one:

Theorem 9. *The natural interval order of all real intervals having length 1 and containing their lower boundaries is a semiorder having its identity as a (strictly* monotonic*!) RIR but having no strictly semiorderlike RIR.*

3.8 Empirical significance?

Our results on ARIRs may be of *mathematical* value only. *Empirical* meaning of real intervals representing two alternatives or experimental stimuli and differing just in one or two real numbers may be difficult to explicate. To make things worse, the results are of no value when A is finite. Otherwise, however, it seems difficult to imagine an economic or psychological theory

[47] Cf. Theorem 9.

[48] [9, Thm. 7.8] states the same theorem without explicitly introducing the notion.

[49] However, we believe that Fishburn's [8, Thm. 10]=[9, Thm. 7.8] *can* be extended to ARIRs and ORIRs in the way one would expect. The proof would require quite a complex modification of Fishburn's respective construction of a weak order on a "doubled" version of A, taking into account lower- and upper-singularity in a similar way as in our suffiency proofs. We withhold this for reasons indicated above.

concerning an infinite set of possible alternatives or stimuli *not* bearing some topology relatively to which borders of real intervals change continuously (cf. Subsection 3.1). With respect to this, ARIRs are strange, while ORIRs (WFPRs) might be comparable to CRIRs (FPRs) (to be investigated).

3.9 Where are the proofs?

This subsection guides from previous claims—in the order of their appearance above—to the places below where they are treated.

Some remarks or clarifications on Theorem 1 are presented in sections 5 and 6. Theorem 2 is proved in Subsection 6.5. Theorems 3 and 4 are proved in Subsection 6.4.

Necessity of representability conditions in theorems 5 and 6 is proved in Section 7. Sufficiency of the same conditions is proved in Subsection 8.5 for Theorem 5 and in Subsection 8.6 for Theorem 6. Concerning (D1) and (D2), we are just reporting [6] without presenting own proofs.[50] Boundedness in both theorems and Condition (*iii*) of the latter are dealt with in Section 5.

Theorem 7 is proved in Subsection 6.6, Theorem 8 in Subsection 6.5. Corollary 1, told to be immediate from the latter theorem, needs nothing more.

Section 5 deals with corollaries 2 and 3. Besides, corollaries 2 through 4 need respective theorems 6, 1, and 5 as well as respective subsections 8.6, 8.2, and 8.5. Corollary 5 needs Corollary 2 and Section 7.

Claims of Subsection 3.7 are treated in Section 9.

The text has been arranged to make each of sections 6–8 work as if the other two were not present. By contrast, most of Section 4 will be needed for each of those sections.

4 Additional general preliminaries.

4.1 Further conventions concerning binary relations.

If $R \subseteq X \times X'$ (and the context exhibits no such sets larger than X, X'), to allow succinct notations we just write Rx' instead of $\{\, x \in X \mid x\ R\ x' \,\}$ and, dually, xR instead of $\{\, x' \in X' \mid x\ R\ x' \,\}$.

We say that some binary relation R ($\subseteq X \times X$, say) has some property *on* a set Y (normally, $Y \subseteq X$) if $R \cap (Y \times Y)$ has that property. Examples will be presented in Subsection 4.2.

R-$\mathrm{Min}\, Y$ will denote the set of R-minima of Y. For asymmetric R—to which we shall apply the notation exclusively—these are elements x of Y such that $y\ R\ x$ for *no* $y \in Y$, or equivalently, such that $x \precsim_R y$ for all $y \in Y$. Dually, R-$\mathrm{Max}\, Y$ will be the set of R-maxima of Y.

[50]Theorem 6 could have been proved using the result of [6] by easily showing "directly", i.e., just in terms of interval orders, equivalence of Condition (*iv*) with (D2).

4.2 Weak orders, their quotients, and representability.

A binary relation R (strictly) *weakly orders* some set X—and (X, R) will then be called a *weak order*—, if R is asymmetric on X and $\precsim_R$ is transitive on X. The latter condition is called *negative transitivity* of R.[51] Recall from Subsection 2.1 that, for R weakly ordering X (as well as for *any* asymmetric binary relation R), $x \precsim_R y$ iff *not* $y\ R\ x$ as well as iff $x\ (R \cup \sim_R)\ y$. It follows that $\sim_R$ is transitive on X (since at least $\sim_R \subseteq \precsim_R \cap \precsim_R^{-1}$) and that, since $\sim_R$ is reflexive and symmetric from the start by its definition and by asymmetry of R, $\sim_R$ is an equivalence relation on X.[52]

We denote the corresponding equivalence class that $x \in X$ belongs to by $[x]_R$. We even shall call it an 'R-equivalence class' (as we did in (F2) already). For $Y \subseteq X$, Y/R will be $\{ [x]_R \mid x \in Y \}$.[53] Moreover, according to the ensuing Lemma 1, a relation R/Y on Y/R is well defined by the condition

$$[x]_R\ R/Y\ [y]_R \quad \text{iff} \quad x\ R\ y \qquad (x, y \in X).$$

Lemma 1. *Assume $R \subseteq X \times X$ weakly orders X. Then $RR \subseteq R$* (R is transitive)*; $R\precsim_R \subseteq R$; and $\precsim_R R \subseteq R$.*

This may be considered "folklore"; though, we briefly present a proof.[54]

Proof. If $x\ R\ y \precsim_R z$, negative transitivity implies $x \precsim_R z$. For $R\precsim_R \subseteq R$, it remains to show that *not* $x \sim_R z$. But $x \sim_R z$ would imply $y \precsim_R z \precsim_R x$ and, by negative transitivity, $y \precsim_R x$, which would (by asymmetry) contradict the assumption $x\ R\ y$.

This yields $R\precsim_R \subseteq R$, and dually obtains $\precsim_R R \subseteq R$.

Finally, $x\ R\ y\ R\ z$ (transitivity, i.e.) is a special case of $x\ R\ y \precsim_R z$ (or of $x \precsim_R y\ R\ z$—as you like). ☐

It may be helpful to notice that $(X/R, R/X)$ is a linear order (if R weakly orders X).[55] A linear order (Q, L) may be viewed as a "reduced" weak order in the sense that L-equivalence classes contain just one element.[56] Conversely, weak orders might be viewed as "redundant" linear orders the equivalence classes of which are occupied by a plurality of elements.—This theme will be continued under the label of 'congruency'.

[51] In this case, namely, the "negation" of R is transitive.

[52] In fact, if R' is any asymmetric binary relation, $\sim_{R'}$ being transitive and being an equivalence relation are equivalent.

[53] To be sure, $\bigcup(Y/R)$ is, in general, no subset of Y.

[54] We are presenting proofs of "trivia", which may appear superfluous to some readers but be helpful for others, in small type.

[55] For deep enlightenment cf. Fishburn's [9, Thm. 1.2], but note that his notation is not as precise as ours.

[56] Compare *categories* where each object is isomorphic to itself only.

In proving Theorems 5f., we shall, like Fishburn [8, 9], use the "Cantor route",[57] or better: the "Birkhoff/Milgram route"[58] to real representability:

Real Homomorphism Theorem. *There is a (strong) real R-homomorphism on X iff R weakly orders X such that X is R-separable.*

This theorem is proved in [7, Thm. 3.1] as well as in [11] or in [17, Subsec. 3.1.4].[59] Concerning the "range of applicability" of the separability notion involved (mentioned in Subsection 2.1 above), the equivalences of definitions according to [3, 1.4.3.f.] are proved there for *weak orders* (X, R), indeed.

4.3 How interval representations handle I, IT, and TI.

The following lemma is an easy consequence of the definitions of I, IT, TI, IRs, and intervals.[60] We leave the proof to the reader.

Lemma 2. *Assume ρ is an IR of (A, T) in a linear order (Q, L) and $a, b \in A$. Then $\rho(a) \cap \rho(b) \neq \emptyset$ iff $a\ I\ b$;[61] if $a\ IT\ b$, then $\rho(a)$ exceeds $\rho(b)$ below; and if $a\ TI\ b$, then $\rho(b)$ exceeds $\rho(a)$ above.*[62]

4.4 Vital features of T, I, IT, and TI.

The ensuing has been presupposed several times earlier in this paper. It is well-known at least since [7] (or see [9, pp. 21f.]).[63]

Lemma 3. *IT and TI weakly order A.*

Though, we present another proof in order to fit the present framework and some needs of later proofs.

In advance we gather, for reference, some other easy observations.

[57] *Fishburn*—[8, p. 93], [9, p. 133]—attributes the following theorem 'essentially' to Cantor's 1895 [4]—cf. [5]. However, there is a trick in proving the necessity of the following representability condition which cannot be seen from this theorem of Cantor's and neither from its proof. Moreover, a problem with the sufficiency part is that Cantor's [4] original theorem uses a separability notion that is strictly stronger than the separability notion used in the "Real Homomorphism Theorem"—cf. the modern, more clearly purely order-theoretic version of Cantor's [4] theorem in [3, 1.5.8] and [3, 1.4.3, 1.4.5].

[58] To continue the previous footnote: according to [17, Subsec. 3.1.4] and [10], the ensuing theorem should better be attributed to G. Birkhoff and A. Milgram.

[59] Also cf. [3, Thm. 1.4.8].

[60] A rigorous proof, however straightforward, would need some lines and cases.

[61] Here, convexity and non-voidness of intervals are essential.

[62] Recall the definitions of Subsection 3.5.

[63] It must have been known to B. Russell according to [18] already, where irreflexivity and transitivity of T and transitivity of IT are assumed, which is another way of defining interval orders.

Lemma 4. *T is irreflexive and*

$$TIT \subseteq T.\text{[64]} \tag{W}$$

I is irreflexive; T is transitive and asymmetric; and the following hold:

$$T \subseteq IT \cap TI; \tag{1}$$

$$\sim_{IT} \cup \sim_{TI} \subseteq I. \tag{2}$$

Proof. Irreflexivity of T and (W) follow from "abstract" interval representability of (A, T) as fixed in Subsection 2.2. E.g., if ρ is an IR and $a\ T\ c\ I\ d\ T\ b$, $\rho(c) \cap \rho(d)$ has (by Lemma 2) an element which "tops" all members of $\rho(a)$ and "is topped" by each member of $\rho(b)$, etc.[65]—Irreflexivity of T implies reflexivity of I by definition of the latter. Therefore, $a\ I\ a\ T\ b$ and $a\ T\ b\ I\ b$, if $a\ T\ b$; which proves (1). The latter yields (2) and, by (W), $TT \subseteq TIT \subseteq T$—so T is transitive and, by irreflexivity, asymmetric. □

Proof of Lemma 3. IT and TI are irreflexive by definition of I. (W) implies $ITIT \subseteq IT$ and $TITI \subseteq TI$, which say that IT and TI are transitive; thus, they are asymmetric, and $\precsim_{IT}, \precsim_{TI}$ are their respective "reverse negations".

The definitions of IT and TI, therefore, imply the equivalences

$$a \precsim_{IT} b \quad \text{iff} \quad Ta \subseteq Tb; \tag{3}$$

$$a \precsim_{TI} b \quad \text{iff} \quad bT \subseteq aT. \tag{4}$$

(3), e.g., obtains thus by contraposition: If $b\ I\ c\ T\ a$, then $c \in Ta \setminus Tb$. If, conversely, $c \in Ta \setminus Tb$, then either $c\ I\ b$, implying $b\ IT\ a$, or $b\ T\ c$, which by transitivity of T (Lemma 4) yields $b\ I\ b\ T\ a$.—(3) and (4) yield transitivity of $\precsim_{IT}$ and $\precsim_{TI}$. □

Now, asymmetry of IT and TI together with (1) adds

$$(IT \cup TI) \cap T^{-1} = \emptyset. \tag{5}$$

4.5 I^*, singularity, and regularity.

This subsection exhibits easy proofs of incidental claims in Subsection 3.6 plus something more needed later on.

[64]In fact, these two conditions were used by N. Wiener [21] to define what is nowadays called 'interval orders'. Nowadays, irreflexivity of T and *if $a\ T\ b$ and $c\ T\ d$, then $a\ T\ d$ or $c\ T\ b$* are common to define the same notion. It is the same notion, indeed: Assuming $b\ I\ c$, the 'common' condition immediately yields (W). If you start at (W) and irreflexivity, $x\ T\ y\ I\ y\ T\ z$ yields transitivity of T, so assuming *not* $c\ T\ b$, the above 'common' condition follows from transitiviy in the case of $b\ T\ c$ and immediately from (W) in the case of $b\ I\ c$.

[65]All of the following is visually clear using ρ—cf. Lemma 2; though we do without to achieve some formal rigour at the cost of relatively little space.

Proposition 1. *$b \in IT$-Max Ia iff $a \in TI$-Min Ib. So $b \in S_-$ iff $I^*b \neq \emptyset$; and $a \in S_+$ iff $aI^* \neq \emptyset$. Moreover, $S_- = \emptyset$ iff $S_+ = \emptyset$.*

The first sentence justifies our definition of I^*.

Proof. By duality, proof of 'only if' suffices for the first sentence. For *reductio*, assume $b \in IT$-Max Ia and $a \notin TI$-Min Ib. The second assumption implies existence of c, d such that $b\ I\ c\ T\ d\ I\ a$, so $b\ IT\ d\ I\ a$. But this contradicts the first assumption, since the latter entails $b\ I\ a$.

The second sentence immediately follows from the definitions and straightforwardly implies the third one. □

Lemma 5. *$a \in S$ iff $a\ I^*\ a$; so $S \subseteq S_- \cup S_+$.*

Proof. For the first statement, we show its contraposition. If *not* $a\ I^*\ a$, there are b, c such that $a\ I\ b\ T\ c\ I\ a$. Hence *not* $b\ I\ c$, so $a \notin S$.—If $a \notin S$, there are $b, c \in Ia$ such that not $b\ I\ c$. Without loss of generality $b\ T\ c$. Then $a\ I\ b\ T\ c\ I\ a$, so—by any one of the two definitions of I^* in Subsection 3.6—*not* $a\ I^*\ a$.

Now the second statement obtains by Proposition 1. □

So far, regularity has been defined "by negation" (via lower-/upper-singularity) only. Here is a "positive" characterizing[66] feature.

Lemma 6. *If $a\ IT\ b$ and $a \notin S_-$, there is an infinite IT-chain in $a(IT) \cap (IT)b$. Dually, if $a\ TI\ b$ and $b \notin S_+$ there is an infinite TI-chain in $a(TI) \cap (TI)b$.*[67]

Proof. $a\ IT\ b$ means that there is some $c_0 \in aI \cap Tb$. If $a \notin S_-$, no $c_n \in aI \cap Tb$ is a TI-minimum of aI, so there is some c_{n+1} such that $a\ I\ c_{n+1}\ TI\ c_n$. By (W), this yields $c_{n+1} \in Tb$. Moreover, there is some $d_n \in A$ such that $c_{n+1}\ T\ d_n\ I\ c_n$.

To conclude, there are infinite sequences (c_n) and (d_n) such that

$$a\ I\ c_{n+2}\ T\ d_{n+1}\ I\ c_{n+1}\ T\ d_n\ I\ c_n\ T\ b$$

for any $n \in \mathbf{N}$. The quintessence of this formula is $a\ IT\ d_{n+1}\ IT\ d_n\ IT\ b$, which meets the first claim concerning some infinite sequence (d_n) in A.

We need not carry out the dual matter. □

4.6 Congruency—exemplified.

If (X, R) is a weak order, a subset Y of X will be called *R-congruent* if it is a union of R-equivalence classes. This is the same as being "closed" with respect to R-equivalence.[68] Some examples needed below are the following.

[66] Lemma 6 claims *one* direction of each characterization only; the proof of the converse direction is easy to see at least from the proof presented.

[67] Actual infinity will never be used; we claim and prove it for narcissistic reasons only.

[68] I.e., Y is R-congruent iff $x \sim_R y \in Y$ implies $x \in Y$.

Lemma 7. *In the same situation, xR and Rx are R-congruent for each $x \in X$. For $a \in A$, aT and $Ia \cap a\precsim_{IT}$ are IT-congruent, while Ta and $Ia \cap \precsim_{TI} a$ are TI-congruent.*

Seemingly everything about weak orders lies clearly visible to the mathematical inner eye. Though, some indications to "formal" proofs might be appropriate.

Proof. If $x\ R\ y' \sim_R y$, then $x\ R\ y$ by Lemma 1.

$a\ T\ b' \sim_{IT} b$ implies $Tb' \subseteq Tb$ by (3) and, therefore, $a\ T\ b$.

Now assume $a\ (I \cap \precsim_{IT})\ b' \sim_{IT} b$. Then, negative transitivity of IT (Lemma 3) yields $a \precsim_{IT} b$. $a\ I\ b$ remains to be proved. For *reductio*, assume $a\ T\ b$ or $b\ T\ a$. In the first case, (3) yields $a\ T\ b'$; while in the second case, (3) yields the $b'\ T\ a$—both contradicting the assumption $a\ I\ b'$.

The remaining proofs are duals of the former ones, using (4) where (A, T) is concerned. ☐

An obvious, though eventually vital feature of congruent sets is

Lemma 8. *If (X, R) is a weak order and $Y \subseteq X$ is R-congruent and non-void, R-$\mathrm{Min}\, Y$ as well as R-$\mathrm{Max}\, Y$ is one R-equivalence class, if not void.*

If we have $R \subseteq X' \times X''$, $R' \subseteq X' \times X'$ such that R' weakly orders X', and $R'' \subseteq X'' \times X''$ such that R'' weakly orders X'', then we shall call R a *strong R'-R''-congruence* if both of the following conditions are satisfied.

(C1) For all $x', y' \in X'$ and $x'', y'' \in X''$ such that $x'\ R\ x''$ and $y'\ R\ y''$

$$x'\ R'\ y' \quad \text{iff} \quad x''\ R''\ y'';$$

(C2)
$$\sim_{R'} R \sim_{R''} \subseteq R.$$

Thus, if R is a *map* $X' \to X''$, (C1) just means that R is a ("strong") (cf. Subsection 2.2) homomorphism respecting R' and reflecting R''; conversely, (C1) generalizes the latter notion from maps to arbitrary binary relations. (C2) tells that R treats R'-equivalent elements the same way, as well as R''-equivalent elements.[69]

(C2) follows from the conjunction of $\sim_{R'} R \subseteq R$ and $R \sim_{R''} \subseteq R$. In proofs, we shall only consider the first part, for the proof of the other one will go dually.

As a particular feature of such relations easily obtains

Lemma 9. *Assume the above situation (so (X', R') and (X'', R'') are weak orders and R is a strong R'-R''-congruence). Then for each $x' \in X'$, $x'R$ is an R''-equivalence class if not void, and $x'R = y'R$ if $x' \sim_{R'} y'$; dually for each $x'' \in X''$, Rx'' is an R'-equivalence class if not void, and $Rx'' = Ry''$ if $x'' \sim_{R''} y''$.*

[69] Of course, identity is implied by the inclusion statement of (C2), since the indifference relations are reflexive.

Proof. If $x' \; R \; x''$ and $x' \; R \; y''$, then $x' \sim_{R'} x'$ and (C1) imply $x'' \sim_{R''} y''$; so $x'R$ is *at most* one R''-equivalence class. By (C2), $x'R$ is, if non-void, *at least* one R''-equivalence class. Again by (C1), it is the same as $y'R$ if neither $x'R$ nor $y'R$ is void and if $x' \sim_R y'$. But if one is not void, by (C2) neither is the other. Dual reasonings yield what remains. □

Thus, if in particular $X' = X''$ and $R' = R''$, R induces a relation R/R' on X'/R' well-defined by the condition

$$[x]_{R'} \; R/R' \; [y]_{R'} \quad \text{iff} \quad x \; R \; y \qquad (x, y \in X').$$

An example for use of this notation will appear in Subsection 4.7.

Another example for strong congruency is I^* (introduced in Subsection 3.6 and reconsidered in Subsection 4.5).

Lemma 10. *I^* is a strong TI-IT-congruence.*

Proof. For (C1), assume $a \; I^* \; b$ and $a' \; I^* \; b'$.

If $a \; TI \; a'$, there is c such that $a \; T \; c \; I \; a'$; therefore $b \; I \; a \; T \; c$. On the other hand, $c \; I \; a' \; I^* \; b'$ implies $c \precsim_{IT} b'$. Putting these consequences together yields $b \; IT \; c \precsim_{IT} b'$. Since IT weakly orders A (Lemma 3), Lemma 1 yields $b \; IT \; b'$.

The converse is proved dually using the dual version of the definition of I^*.

For (C2), conclude $a' \in TI\text{-Min}\, Ib = TI\text{-Min}(Ib \cap \precsim_{TI} b)$ from $a \sim_{TI} a' \; I^* \; b$ to get $a \; I^* \; b$ by lemmas 7f. □

As a consequence, we get another example of congruency of sets.

Lemma 11. *S_- is TI-congruent; dually, S_+ is IT-congruent.*

Proof. For the first of the dual parts assume $a \in S_-$ and $a \sim_{TI} b$. By lemmas 9 and 10, then, $I^*a = I^*b$. Therefore $b \in S_-$, as well. □

As a further example of strong congruency we define a relation T^* on A by

$$T^* := T \setminus TIT.$$

This is how Fishburn [8, p. 95] started to present his version of Condition (F2). Note that, by (W), $a \; T^* \; b$ iff $a \in TI\text{-Max}\, Tb$ and iff $b \in IT\text{-Min}\, aT$.[70] So if (A, T) is *natural* (Subsection 2.2), $a \; T^* \; b$ means that a, b "touch" each other in the sense that a is one of the most close intervals below b and—equivalently—that b is one of the most close intervals above a.

Now we continue the previous findings by

Lemma 12. *T^* is a strong TI-IT-congruence.*

Proof. For (C1), assume $a \; T^* \; b$ and $a' \; T^* \; b'$. If $a \; TI \; a'$, there is c such that $a \; T \; c \; I \; a'$. By $a \; T^* \; b$, this implies $b \precsim_{IT} c$; while by $a' \; T^* \; b'$ (and $T^* \subseteq T$), it implies $c \; IT \; b'$. By Lemma 1 both consequences together yield $b \; IT \; b'$. (Conversely dually.)

(C2) obtains, similarly to the above, by reading $a' \; T^* \; b$ as $a' \in TI\text{-Max}\, Tb$ and from lemmas 7f. □

[70] [20, p. 59, (19)] seems to overlook that the latter two propositions are equivalent.

4.7 "Neighbourhood", "jumps", and regularity.

For any R weakly ordering some set X, we define an associated relation

$$N_R := R \setminus RR.$$[71]

$x\ N_R\ y$ means that x and y are *immediate R-neighbours* (in "direction" $x\ R\ y$, to be precise), and might have been defined equivalently by $x \in R\text{-Max}\, Ry$ or by $y \in R\text{-Min}\, xR$.

Lemma 13. *In the situation just depicted, N_R is a strong R-R-congruence.*

Proof. For (C1), assume $x'\ N_R\ x''$ and $y'\ N_R\ y''$. If $x'\ R\ y'$, then $x'' \precsim_R y'$ by $x'' \in R\text{-Min}\, x'R$; and Lemma 1 using $N_R \subseteq R$ yields $x''\ R\ y''$. The converse is proved dually by using $y' \in R\text{-Max}\, Ry''$.

For (C2), interpret $a \sim_R a'\ N_R\ b$ as $a \sim_R a' \in R\text{-Max}\, Rb$. $a\ N_R\ b$ then obtains by lemmas 7f. □

We call each element of N_R/R an *R-jump of X*. By the above "Real Homomorphism Theorem", representability of (X, R) by a "utility function" requires that there are only countably many R-jumps of X—cf. Lemma 15 below.

The following concerns regularity.

Lemma 14. *If $a \in A \setminus S_-$, then $aN_{IT} = \emptyset$.*

Proof. Assume $a \notin S_-$ and, for *reductio*, $a\ N_{IT}\ b$ for some $b \in A$. This yields $a\ IT\ b$. By Lemma 6 and the first assumption, there is some $c \in A$ such that $a\ IT\ c\ IT\ b$, contradicting the second assumption. □

4.8 Miscellanea on separability.

We insert a step into the definition of separability, a step we omitted in Subsection 2.1. If R is some binary relation on a set X, a set D will be called *R-dense in X* if $D \subseteq X$ and if for all $x, y \in X$ such that $x\ R\ y$ there is $d \in D$ such that $x \precsim_R d \precsim_R y$. Thus, X is R-separable iff there is a countable set that is R-dense in X. We are now in a position to state, for reference, the obvious

Lemma 15. *Let R weakly order X and D be R-dense in X. Then, for every R-jump (B, C) of X, $(B \cup C) \cap D$ is non-void. If X is R-separable, therefore, it has only countably many R-jumps.*

Furthermore we shall need:

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[71] For more preciseness, the term should also specify a *range*, since the latter determines "neighbourhood" of two objects as well as the relation R. (In the situation depicted, new "neighbourhoods" of two objects my arise by removing some others between them.) Nevertheless, we drop the range and shall indicate it by the context.

Lemma 16. *If (X, R) is a weak order such that X is R-separable, and if $Y \subseteq X$, then Y is R-separable.*

Proof. Assume the hypothesis. By the Real Homomorphism Theorem, then, there is a real R-homomorphism on X. It is a real R-homomorphism on $Y \subseteq X$, as well. Therefore, the Real Homomorphism Theorem entails that Y as well is R-separable.[72] □

5 Proofs: "Unbounded"/FP representations.

This section provides the obvious (if so: skippable) explanations

1. of the equivalences of the first two conditions in each of Theorem 1 and of Theorems 5f., resp., (where irrelevance of the boundedness restriction for the existence of RIRs of the repective kinds is claimed);
2. concerning the existence of WFPRs in Theorem 6;
3. concerning exactness, singular-exactness and what corresponds to them for FPRs or WFPRs according to corollaries 2 and 3.

A CBRIR *is* a CRIR, an ABRIR is an ARIR, an OBRIR is an ORIR (thus in each case, existence of a former implies existence of a latter). Concerning the converse existence claims:

As may be recalled from trigonometry or from elementary analysis, arctan is a strictly monotone (therefore one-to-one) map from $\mathbf{R}$ onto the open real interval $(-\pi/2, \pi/2)$.[73] Now, if F is some ARIR or some ORIR of (A, T),

$$a \mapsto \arctan[F(a)] \qquad (a \in A)$$

is an ABRIR or an OBRIR of (A, T), respectively.[74]—If F is, instead, a CRIR, we proceed essentially in the same way, but we complete $\arctan[F(a)]$ by each of $-\pi/2, \pi/2$, resp., whenever one of them is "touched" by $\arctan[F(a)]$. More precisely, we define, for every $a \in A$, $l(a)$ to be $\emptyset$ if $F(a)$ is bounded; to be $\{\pi/2\}$ if $F(a)$ has a lower, but no upper bound; to be $\{-\pi/2\}$ if $F(a)$ has an upper, but no lower bound; finally to be $\{-\pi/2, \pi/2\}$ if $F(a)$ is unbounded in each direction. Then

$$a \mapsto \arctan[F(a)] \cup l(a) \qquad (a \in A)$$

is a CBRIR of (A, T).—This completes proofs of all the earlier claims of irrelevance of boundedness restrictions for existence of RIRs.

Next (concerning theorems 1 and 6), by analogy to Subsection 3.1, if F is an OBRIR, $u : A \to \mathbf{R}, a \mapsto \inf F(a)$ and $v : A \to \mathbf{R}, a \mapsto \sup F(a)$ form a WFPR; and if (u, v) is a WFPR, $a \mapsto (u(a), v(a))$ is an OBRIR.[75] Thus, as claimed in Theorem 6, existence of a WFPR is the same as existence of an OBRIR. The analogous equivalence in Theorem 1 was explained in Subsection 3.1 already.

[72] The fact may also be proved elementarily on less than a whole page.

[73] It is a map inverse to one "branch" of $\tan = \sin / \cos$. Of course, any other strictly monotone map from $\mathbf{R}$ into a bounded real interval would do as well.

[74] Each real interval occurring has $-\pi/2$ as a lower and $\pi/2$ as an upper bound.

[75] According to widespread use, $(u(a), v(a))$ here denotes $\{\, r \in \mathbf{R} \mid u(a) < r < v(a) \,\}$, of course.

Finally (concerning corollaries 2 and 3), it is easy to check that a CBRIR or OBRIR is exact iff for the corresponding (according to Subsection 3.1) FPR or WFPR (u, v), u is a real IT-homomorphism and v is a real TI-homomorphism (in each case, one direction needs Lemma 2). It is obvious that a CBRIR is singular-exact iff for the corresponding FPR (u, v), $u(b) = v(a)$ iff $a\ I^*\ b$. This proves equivalence of the last two conditions of corollaries 2 and 3.

6 Proofs on representability conditions.

6.1 Dual conditions in general.

While R-separability is a self-dual notion for any relation R, neither (F2), nor (E), nor (G) are (at least "syntactically") self-dual.[76] Though, in their contexts, they may be replaced by their duals.

To make this clear, we begin by introducing some shorthand definitions. If J is a set of real intervals, let J^* be the image of J under multiplying by -1 (e.g., $[p, q)$ turns into $(-q, -p]$). We say ρ is a RIR of some interval order (X, R) in some set J of real intervals if ρ is an IR of (X, R) in $(\mathbf{R}, <)$ such that $\rho[X] \subseteq J$. By multiplying the members of the image intervals by -1, we obtain some other map ρ^* from X into the set of real intervals. The *dual* of an interval order (X, R) is (X, R^{-1}). Observe that ρ is a RIR of some interval order $\mathcal{T}$ in some set J of real intervals iff ρ^* is a RIR of the dual of $\mathcal{T}$ in J^*. Observe, furthermore, that some interval order meets some condition iff the dual interval order meets the dual condition.

Now we may state and prove:

Theorem 10. *Let J be some set of real intervals such that $J^* \subseteq J$. Let (C) be some condition on interval orders necessary and sufficient for existence of a RIR in J. Then the dual of (C) is necessary and sufficient for existence of an RIR in J, too.*

Proof. Assume the hypotheses, and let $\mathcal{T}$ be some interval order.

For necessity of the dual of (C), assume $\mathcal{T}$ has a RIR ρ in J. Then ρ^* is a RIR of the dual $\mathcal{T}^*$ of $\mathcal{T}$ in J^* and, by hypothesis, in J. By the other hypothesis, $\mathcal{T}^*$ meets (C). The dual $\mathcal{T}$ of $\mathcal{T}^*$, therefore, meets the dual of (C).

For suffiency of the dual of (C), assume $\mathcal{T}$ meets the dual of (C). Then, the dual $\mathcal{T}^*$ of $\mathcal{T}$ meets (C). By hypothesis, $\mathcal{T}^*$ has a RIR ρ in J. ρ^*, then, is a RIR of $\mathcal{T}$ in J^*. By hypothesis, it is a RIR in J, as well. $\square$

Admittedly, this is no strict mathematical proof. Even the 'theorem' is too vague ('condition on interval orders') to be a clear mathematical proposition. A clear and strict mathematical version would need to introduce a formal language and to present model-theoretical statements.

[76] Indeed it is, presumably, not too difficult to find interval orders where one of these conditions holds but not its dual. I.e., the conditions are even "semantically" not self-dual.

When we apply the 'theorem' to (F2), (E), and (G) (and to their duals), however, it suffices that the previous indicates how to derive the dual of each of them in their context. Viewing J from Theorem 10 as the set of *closed* real intervals leads to the following

Corollary 9. *In Theorem 1, the dual of (E)*[77] *may be added to the list of equivalent conditions, and (F2) may be replaced by its dual. In Theorem 7, (G) may be replaced by its dual.*

Proof. The case of (E) is straightforward. For (F2) and (G), one uses that (F1) is self-dual. □

While the latter example of Theorem 10 is related to the notion of a CRIR, the other RIR notions introduced earlier lead to further obvious examples of J from Theorem 10.

All the above may have been "folklore" and arises from the "symmetry" of $(\mathbf{R}, <)$; i.e., $(\mathbf{R}, >)$ is isomorphic to $(\mathbf{R}, <)$.

6.2 A preliminary on (F2).

We supply a name for the set (F2) is about. S will denote the set of singular elements of A in Fishburn's sense. By lemmas 9 and 12, $aT^* = bT^*$ if $a \sim_{TI} b$. Therefore, we are allowed to define

$$G_0 := \{\, [a]_{TI} \mid a \in A;\ S \cap [a]_{TI} = \emptyset;\ aT^* \neq \emptyset;\ S \cap aT^* = \emptyset \,\}.$$

Thus, (F2) tells G_0 to be countable.[78]

6.3 Condition (E).

Condition (E) appears in Theorem 1 just to summarize earlier contributions, relying on the equivalences of (F1)&(F2) with existence of a CBRIR as found by Fishburn and of (E) with existence of a FPR as found by [15]. After Section 5, the claim of Theorem 1 concerning (E) then follows from the obvious correspondence between FPRs and CBRIRs depicted in Subsection 3.1.

[77] The dual of (E) is: *there be a countable $D \subseteq A$ such that for all $(a,b) \in T$ there is some $d \in D$ such that $a \precsim_{TI} d\ T\ b$.* To check that this is the dual of (E), dualize the components $a\ T\ b$, $a\ T\ d$, and $d \precsim_{IT} b$ of (E), then interchange a, b.

[78] Fishburn [8] originally defined an equivalence relation E on T^* such that $(a,b)\ E\ (a',b')$ iff $a\ T\ b'$ and $a'\ T\ b$. Now, this is just equivalent to $a \sim_{TI} a'$: Keep in mind that $a\ T^*\ b$ and $a'\ T^*\ b'$. Then $a \sim_{TI} a'$ implies $a\ T\ b'$ and $a'\ T\ b$ by (4). Conversely, prove the contraposition. If $a\ TI\ a'$, say $a\ T\ c\ I\ a'$. Then $b \precsim_{IT} c$ by $a\ T^*\ b$. Therefore, by (3), $Tb \subseteq Tc$, so the assumption $c\ I\ a'$ excludes $a'\ T\ c$ and $a'\ T\ b$.—Denoting the set of equivalence classes of T^* with respect to E by T^*/E, countability of $G_0' := \{\, P \in T^*/E \mid P \cap (S \times A) = \emptyset = P \cap (A \times S) \,\}$ comes very close to how Fishburn [8] presented (F2). But by the equivalence of the conditions noted above, $P \mapsto pr_1[P]$ is a one-to-one map from G_0' onto G_0, where pr_1 denotes the projection to the first factor.

In [9], Fishburn uses another version of G_0, which would need too much preparations to be explained in the present framework.

The previous, however, is not the whole truth. Whereas interval orders usually are considered asymmetric and therefore irreflexive, [15] deal with "weak", i.e., reflexive "interval orders". Furthermore, it remains to check our claim that Condition (E) presented here is equivalent to what in [15] is defined to be 'i. o.-separable'. Summarizing, it remains to check whether Subsection 3.1 presents a correct "translation" of [15].[79]

These tasks are not difficult. But if we rely on Fishburn's [8, 9] and on having correctly reported them, we may replace such a check by a proof of the following just in terms of (A, T), which may be of some value on its own right.

We begin with some observations concerning singularity.

Lemma 17. *If $a\ T^*\ b$ and $a \sim_{TI} c \in S$, then $c\ N_{IT}\ b$. Dually, if $a\ T^*\ b$ and $b \sim_{IT} c \in S$, then $a\ N_{TI}\ c$.*

Proof. By (2) (Lemma 4), $a \sim_{TI} c \in S$ implies $a\ I\ c$. By $a\ T^*\ b$ obtains $c\ I\ a\ T\ b$, thus $c\ IT\ b$.

Now, for *reductio*, assume $c\ I\ d'\ T\ d\ IT\ b$ for some $d, d' \in A$. By Lemma 12, $c\ T^*\ b$, i.e., $b \in IT\text{-}\mathrm{Min}\, cT$, so *not* $c\ T\ d$. $d\ T\ c$ is ruled out by (5) and the assumption. We conclude $c\ I\ d$.

Using the assumptions about d', we get $d'\ I\ c\ I\ d$. By $c \in S$, then, obtains $d'\ I\ d$, which contradicts the assumption $d'\ T\ d$. Therefore, $c\ ITIT\ b$ does *not* hold. $\square$

Lemma 18. *Let $c, d \in S$. Then $c \sim_{IT} d$ iff $c \sim_{TI} d$.*

Proof. Assume $c \sim_{IT} d$. (2) yields $c\ I\ d$. If $c\ T\ a\ I\ d$ for some $a \in A$, then $a\ I\ d\ I\ c$ and, by singularity, $a\ I\ c$—contradicting $c\ T\ a$. The assumption $d\ T\ a\ I\ c$ is rejected by interchanging c and d in the former reasoning. Together, this yields $c \sim_{TI} d$. The converse direction is proved dually. $\square$

We can now prove the main claim of the present subsection.

Proposition 2. *(E) holds iff (F1) and (F2) hold.*

Proof. Let us firstly assume (F1) and (F2) in order to derive (E). So by TI-separability of A and by lemmas 3 and 15, there are only countably many TI-jumps. We may, therefore, presuppose a countable $D_+ \subseteq A$ which intersects with $B' \cap S$ whenever there is a TI-jump (B, B'). By IT-separability, there is a countable $D_- \subseteq A$, and, similarly to the previous reasoning, we may assume that D_- intersects with the right-hand term of each IT-jump. Furthermore, by lemmas 9 and 12 (strong congruency of T^*), for every $B \in G_0$ we may pick d_B from A such that for all $b \in B$, $b\ T^*\ d$. Let D_0 be the countable set $\{\, d_B \mid B \in G_0 \,\}$. Assume $(a, b) \in T$. We show that the countable set $D_- \cup D_+ \cup D_0$ contains d such that $a\ T\ d \precsim_{IT} b$.

[79] One firstly has to check that $\precsim_T = T \cup I$ is an "interval order" in the sense of [15]. Assuming this, [15] define sets $B(a, b) := \{\, d \in A \mid \text{there is } c \in A \text{ such that } a \precsim_T c\ T\ d \precsim_T b \,\}$. (In fact, [15] write A where we write B for having used A for another purpose already.) Now one has to check that the condition $a\ T\ d \precsim_{IT} b$ being part of our (E) is equivalent to the original condition $d \in B(a, b) \setminus (B(a, a) \cup B(b, b))$.

If *not* $a\,T^*\,b$, there is c such that $a\,T\,c\,IT\,b$. Then by IT-separability, there is $d \in D_-$ such that $c \precsim_{IT} d \precsim_{IT} b$. By (3) obtains $a\,T\,d \precsim_{IT} b$.

If $a\,T^*\,b$, the way of reasoning depends on whether $[a]_{TI}$, $[b]_{IT}$, or none of both contain a singular element.

If there is $c \in S \cap [a]_{TI}$, Lemma 17 yields $c\,N_{IT}\,b$. By the additional assumption on D_-, the latter set contains some $d \sim_{IT} b$. By strong congruency of T^* obtains $a\,T\,d \precsim_{IT} b$.

If there is $c \in S \cap [b]_{IT}$, Lemma 17 yields $a\,N_{TI}\,c$, and D_+ has been chosen to contain some *singular* $d \sim_{TI} c$. By Lemma 18, $c \sim_{IT} d$. Furthermore, $c \sim_{IT} b$, yielding $d \sim_{IT} b$. Now (3) yields $a\,T\,d \precsim_{IT} b$.

Finally, if $S \cap ([a]_{TI} \cup [b]_{IT}) = \emptyset$, strong congruency of T^* implies $S \cap aT^* = \emptyset$. Therefore $[a]_{TI} \in G_0$, and there is some $d \in D_0 \cap aT^*$. Strong congruency of T^* now implies $d \sim_{IT} b$, and this yields $a\,T\,d \precsim_{IT} b$ again.

For the other direction, we now assume (E) in order to derive (F1) and (F2). So let $D \subseteq A$ be countable such that for all $(a, b) \in T$ there is $d \in D$ such that $a\,T\,d \precsim_{IT} b$.

For (F2), we show that there is a one-to-one map $B \mapsto d_B$ from G_0 into D. If $B \in G_0$, pick b from B and a from bT^*. Now we may pick d_B from D such that $b\,T\,d_B \precsim_{IT} a$.[80] $b\,T^*\,a$ implies $d_B \sim_{IT} a$. If $B' \neq B$ is another element of G_0, Lemma 12 by way of (C1) yields that $d_{B'}$ is not IT-equivalent to d_B. Therefore, $d_B \neq d_{B'}$.

For the IT-separability part of (F1), assume $a\,I\,c\,T\,b$. Then $c\,T\,d \precsim_{IT} b$ for some $d \in D$, and, therefore, $a \precsim_{IT} d \precsim_{IT} b$.

Finally, TI-separability follows dually to the previous from the dual of (E), which holds according to Corollary 9. □

Instead of proving TI-separability above, we could, by Theorem 11 below, just have proved that there are only countably many TI-jumps.[81] Indeed, *this* is a way to meet our announcement of proving Proposition 2 'just in terms of (A, T)'—whereas the last sentence of the previous proof draws on Corollary 9, which was *not* proved 'just in terms of (A, T)'.

6.4 Singular-separability and -countability.

Proof of theorems 3 and 4. By Proposition 1, $S_- = \emptyset$ iff $S_+ = \emptyset$.

Otherwise, by strong congruency of I^* (lemmas 9 and 10), the latter induces a one-to-one map Σ_- from S_+/TI onto S_-/IT, *viz.* $\Sigma_-([a]_{TI}) = [b]_{IT}$ iff $a\,I^*\,b$. This proves Theorem 4, already.

By (C1) as applied to I^*, Σ_- is an isomorphism from $(S_+/TI, TI/S_+)$ onto $(S_-/IT, IT/S_-)$ (recall notation from Subsection 4.2). Therefore, S_+/TI is TI/S_+-separable iff S_-/IT is IT/S_--separable. Furthermore, S_+/TI is TI/S_+-separable iff S_+ is TI-separable, and the obvious dual holds for IT-separability of S_-. This proves Theorem 3. □

[80] Independence on the choice of b holds in many respects, but is not needed here.

[81] If $(B, B') = ([b]_{TI}, [b']_{TI})$ is a TI-jump such that $b\,T\,a\,I\,b'$ for some $a \in A$, there is $d_B \in D$ such that $b\,T\,d_B \precsim_{IT} a$. Indeed, $b\,T^*\,d_B$, because $b\,N_{TI}\,b'$. By strong congruency of T^*, $d_C \neq d_B$ if $(C, C') \neq (B, B')$ is another TI-jump. Therefore, there can be only countably many TI-jumps.

6.5 (F1), above, and below.

This subsection proves theorems 2 and 8; moreover, it proves, meeting the announcement below Theorem 1, equivalence to (F1) of two "practically weaker" versions.

Proof of Theorem 2. Assume (A, T) is a semiorder. As Fishburn [8, 9, Sec. 7.5] notes for deriving (F1) from T_0-separability, any (countable) set T_0-dense in A is IT- and TI-dense in A, as well. But the converse direction is hardly noticeably less obvious: If D is (countable and) IT-dense in A and D' is (countable and) TI-dense in A, $D \cup D'$ is (countable and) T_0-dense in A.[82] $\square$

Proof of Theorem 8. Firstly assume Condition (i) to derive (F1). Since Condition (i) is self-dual, we confine ourselves to proving that (A, T) is IT-separable. By the assumption of singular-countability and regular-separability, there is a countable $D \subseteq A$ which intersects with each $[a]_{IT}$ where $a \in S_-$ and which for every $a, b \in A \setminus S_-$ such that $a\ IT\ b$ contains some d such that $a \precsim_{IT} d \precsim_{IT} b$.[83] Now assume just $a\ IT\ b$. If $a \in S_-$, there is an IT-equivalent $d \in D$. If not, Lemma 6 tells that there is some $c \in a(IT) \cap (IT)b$. If $c \in S_-$, there is an IT-equivalent $d \in D$. If not, D contains some d such that $a \precsim_{IT} d \precsim_{IT} c$. In each case, $a \precsim_{IT} d \precsim_{IT} b$.

The implication from (F1) to Condition (iii) derives from Lemma 16.

Now, we need examples refuting the converse directions: Firstly, if (A, T) is the natural interval order of all closed bounded real intervals, (F1) is met, but not Condition (i). Indeed, in this case $A = S_- = S_+$, so (A, T) is not singular-countable, since the real numbers correspond one-to-one to the IT- as well as to the TI-equivalence classes. (F1) is met since this correspondence even gives rise to an isomorphism from $(A/IT, IT/A)$ onto $(\mathbf{R}, <)$ as well as from $(A/TI, TI/A)$ onto $(\mathbf{R}, >)$.

If (A, T) is the natural interval order of *all* bounded real intervals, Condition (iii) is met, but not (F1).[84] Here, "open ends" indicate belonging to $A \setminus S_-$ or $A \setminus S_+$, while "closed ends" indicate belonging to S_- or S_+. In each of these four cases, there is an isomorphism onto $(\mathbf{R}, <)$ or onto $(\mathbf{R}, >)$ much like before, therefore Condition (iii) holds. On the other hand, $(A/IT, IT/A)$ (e.g.) is now isomorphic to the lexicographic order on $\mathbf{R} \times \{0, 1\}$, which has one jump for each real and therefore is not separable according to Lemma 15.[85] $\square$

[82] The claim of Theorem 2, thus, is so obvious that it deserves being mentioned only because *Fishburn* seems to have overlooked it.

[83] In fact, $d \in A \setminus S_-$, but this will not matter.

[84] This is just another, somewhat more technical, presentation of the example from Subsection 3.3.

[85] Two rather "unnatural" (but more explicit and "parsimonious") examples serving, in view of our Theorem 5, for the same purpose have been presented by Fishburn in [8, 9, Sec.s 7.5f.].

Here come the announced “weak” versions of (F1).

Theorem 11. *The following conditions are equivalent.*

(i) *(F1) holds;*

(ii) *A is IT-separable and has only countably many TI-jumps;*

(iii) *A is TI-separable and has only countably many IT-jumps.*

Proof. (F1) implies the other two conditions by Lemma 15. By duality, it remains to conclude (e.g.) from Condition (ii) that A is TI-separable.

Thus, assume Condition (ii). There is, then, a countable D being IT-dense in A as well as a countable $D_0 \subseteq A$ intersecting with every $[a]_{TI}$ where $aN_{TI} \neq \emptyset$. For every $(d, d') \in (D \times D) \cap IT$, pick one $c_{d,d'}$ from A such that $d\ I\ c_{d,d'}\ T\ d'$. Let D_1 be the countable $\{\, c_{d,d'} \mid d, d' \in D;\ d\ IT\ d' \,\}$. For convenience, we aim at *reductio* and assume that the countable $D_0 \cup D_1 \subseteq A$ is *not* TI-dense in A.

If so, there is some $(a, b) \in TI$ such that $a \precsim_{TI} d \precsim_{TI} b$ for *no* $d \in D_0 \cap D_1$. Therefore (think of D_0), $aN_{TI} = \emptyset$, and there are $a_0, a_1, a_2 \in A$ such that $a\ TI\ a_0\ TI\ a_1\ TI\ a_2\ TI\ b$. This forces existence of “witnesses” b_0, b_1, b_2, b_3 such that

$$a\ T\ b_0\ I\ a_0\ T\ b_1\ I\ a_1\ T\ b_2\ I\ a_2\ T\ b_3\ I\ b.$$

There are $d_0, d_2 \in D$ such that

$$b_0 \precsim_{IT} d_0 \precsim_{IT} b_1\ IT\ b_2 \precsim_{IT} d_2 \precsim_{IT} b_3.$$

By Lemma 1, $d_0\ IT\ d_2$; so there is $d \in D_1$ such that $d_0\ I\ d\ T\ d_2$. By (3), $a\ T\ d_0\ I\ d\ T\ b_3\ I\ b$; thus, $a \precsim_{TI} d \precsim_{TI} b$—contradicting the assumption. $\square$

6.6 “Weakening” Fishburn’s Condition (F2).

This subsection proves Theorem 7.

As G_0 already served to reformulate (F2), we define a variant of it to reformulate (G):

$$G_1 := \{\, [a]_{TI} \mid a \in A;\ S_+ \cap [a]_{TI} = \emptyset;\ aT^* \neq \emptyset;\ S_- \cap aT^* = \emptyset \,\}.$$

Thus, (G) tells G_1 to be countable.

The definition of G_1 differs to that of G_0 in replacing S by S_+ at one place and by S_- at another. Since $S \subseteq S_- \cap S_+$ (Lemma 5), therefore, $G_1 \subseteq G_0$.

By Proposition 1, however, $S_+ \cap [a]_{TI} = \emptyset$ iff $a \notin S_+$. Thus, G_1 may be rewritten in the following way.

$$G_1 = \{\, [a]_{TI} \mid a \in A \setminus S_+;\ aT^* \neq \emptyset;\ S_- \cap aT^* = \emptyset \,\}.$$

By Lemma 11, $S_+ \cap [a]_{TI} \neq \emptyset$ iff $a \in S_+$. The difference set $G_0 \setminus G_1$, therefore, is

$$\begin{aligned} G_2 := \{\, [a]_{TI} \mid a \in S_+;\ & aT^* \neq \emptyset; \\ & S \cap [a]_{TI} = \emptyset; \\ & S \cap aT^* = \emptyset \neq S_- \cap aT^* \,\}. \end{aligned}$$

We shall compare this to the further variants

$$\begin{aligned} G_3 &:= \{\, [a]_{TI} \mid a \in S_+;\ aT^* \neq \emptyset \,\}; \\ G_4 &:= \{\, [a]_{TI} \mid a \in A;\ aT^* \neq \emptyset;\ S_- \cap aT^* \neq \emptyset \,\}. \end{aligned}$$

Proof of Theorem 7. Theorem 7, now, just tells that, if (F1), G_0 is countable iff G_1 is countable. Being a subset of G_0, G_1 is countable if G_0 is countable without recourse to (F1). Thus, it suffices to prove that, if (F1) and if G_1 is countable, G_0 is countable, as well. And this just amounts to showing that, if (F1), G_2 is countable. Now, $G_2 \subseteq G_3 \cup G_4$, so it suffices to realize that, if (F1), G_3 and G_4 are countable.

Concerning G_3: Iff $[a]_{TI} \in G_3$, there are $b, c \in A$ such that $a\ T^*\ b$ and $a\ I^*\ c$. In this case, $c\ N_{IT}\ b$. Strong congruency of T^* and N_{IT} (lemmas 9, 12, 13), therefore, induces a one-to-one map $[a]_{TI} \mapsto (N_{IT}(aT^*), aT^*)$[86] from G_3 into (even onto) the set of IT-jumps of A. Now, by (F1) and Lemma 15, the set of these jumps is countable.

Concerning G_4: Iff $[a]_{TI} \in G_4$, there are $b, c \in A$ such that $a\ T^*\ b$ and $c\ I^*\ b$. In this case, $a\ N_{TI}\ c$. Therefore, by strong congruency of N_{TI}, there is a one-to-one map $[a]_{TI} \mapsto ([a]_{TI}, aN_{TI})$ from G_3 into (even onto) the set of TI-jumps of A. Now, by (F1) and Lemma 15, the set of these jumps is countable. □

7 Necessity proofs.

In this section we prove that each of the representability conditions (*iii*) in Theorem 5 and (*iv*) in Theorem 6 follows from one of the other conditions listed in the respective theorem. Indeed, by Section 5, then, in each case the representability condition is necessary for *all* the other conditions of the respective theorem.—Finally, we nearly prove Corollary 5, only ignoring its "component" concerning existence of O(B)RIRs.

The following just summarizes a proof technique from Fishburn's [8].

We call φ a *weak real R-homomorphism on X* if it maps X into $\mathbf{R}$ and R is some binary relation such that $x\ R\ y$ (just) implies $\varphi(x) < \varphi(y)$. Thus in contrast to a strong one, a weak real R-homomorphism may treat R-equivalent elements differently. Now the following differs from the "Real Homomorphism Theorem" (Subsection 4.2) in but one word.

[86] $N_{IT}(aT^*)$ here denotes $\{\, c \in A \mid c\ N_{IT}\ b \in aT^* \text{ for some } b \,\}$.

Real Homomorphism Corollary. *There is a* weak *real R-homomorphism on X iff R weakly orders X such that X is R-separable.*

Proof. If R weakly orders X such that X is R-separable, by the Real Homomorphism Theorem there is a strong real R-homomorphism on X, which *a fortiori* is a *weak* real R-homomorphism on X. If there is a weak real R-homomorphism φ on X, define $x \ R' \ y$ by $\varphi(x) < \varphi(y)$ for $x, y \in X$. Then φ is a strong real R'-homomorphism, and by the Real Homomorphism Theorem there is a countable D being R'-dense in X. Moreover, $R \cap (X \times X) \subseteq R'$, and if $x, y \in X$ such that $x \precsim_{R'} y$, then $x \precsim_R y$. Therefore, D is R-dense in X, as well. □

The following completes necessity of Condition (*iii*) in Theorem 5 and yields one part of necessity of Condition (*iv*) in Theorem 6.

Proposition 3. *If (A, T) has a RIR, it is singular- and regular-separable.*

Proof. Let ρ be some RIR of (A, T). We first show that S_- is IT-separable, ignoring the dual implicit claim. For any $a \in S_-$, pick b_a from I^*a and q_a from $\rho(b_a) \cap \rho(a)$ (Lemma 2). This amounts to a map $\varphi : S_- \to \mathbf{R}, a \mapsto q_a$. Now if $a_0, a_1 \in S_-$ and $a_0 \ IT \ a_1$, lower-singularity yields $b_{a_0} \ T \ a_1$, hence $\varphi(a_0) < \varphi(a_1)$. To sum up, φ is a weak real IT-homomorphism on S_-, so by the Real Homomorphism Corollary the latter is IT-separable.

Now we show that $A \setminus S_-$ is IT-separable and, again, omit the dual claim. By Section 5, we may assume that ρ is a BRIR. Then $\varphi : a \to \inf \rho(a)$ maps $A \setminus S_-$ into $\mathbf{R}$. If $a, b \in A \setminus S_-$ and $a \ IT \ b$, there is c such that $a \ IT \ c \ IT \ b$ by Lemma 6. By Lemma 2, there are $p \in \rho(a)$ and $q \in \rho(c)$ such that $p < q < r$ for all $r \in \rho(b)$. Hence $\varphi(a) \leq p < q \leq \varphi(b)$ and, indeed, $\varphi(a) < \varphi(b)$. So this φ is a weak real IT-homomorphism on $A \setminus S_-$, and we conclude that $A \setminus S_-$ is IT-separable using the Real Homomorphism Corollary. □

The following completes necessity of Condition (*iv*) in Theorem 6.

Proposition 4. *If (A, T) has an ORIR, it is singular-countable.*

Proof. Let ρ be an ORIR. Let S' be a subset of S_- which intersects exactly once with each element of S_-/IT. For each $a \in S'$ pick one b_a from I^*a. Then $J := \{ \rho(b_a) \cap \rho(a) \mid a \in S' \}$ is (by Lemma 2 and lower-singularity) a set of pairwise disjoint open real intervals. The countable set of rational numbers intersects with every open real interval, so J, hence S', and hence S_-/IT is countable. By Theorem 4 (proved in Subsection 6.4 already) S_+/TI, then, is countable, as well. □

Now, concerning Corollary 5:

Proposition 5. *An ORIR of (A, T) is singular-exact iff $S_- = \emptyset$ iff $S_+ = \emptyset$.*

Proof. $S_- = \emptyset$ iff $S_+ = \emptyset$ iff $I^* = \emptyset$—cf. Proposition 1. Now, if $I^* = \emptyset$, any ORIR is trivially singular-exact. On the other hand, if ρ is an ORIR and $a\ I^*\ b$, then $\rho(a) \cap \rho(b)$ is (by Lemma 2) an *open* (non-void) interval and, hence, contains an infinity of points. So if ρ is singular-exact, then $I^* = \emptyset$. □

8 Sufficiency proofs.

8.1 Content and general strategy.

This section completes proofs of theorems 5 and 6 as well as of corollaries 2 through 5 (subsections 3.3, 3.5, and 3.6). There is the "closed", the "arbitrary", and the "open" case. In each case, one of theorems 1, 5, and 6 together with one or two of the corollaries mentioned forms a more comprehensive equivalence proposition. More precisely, in each case a number of countability conditions like (E) and the conditions of Theorem 8 is claimed to be equivalent to several claims of existence of a RIR. Of these existence claims, the ones from subsections 3.5 and 3.6 concerning exactness and singular-exactness, are "logically" stronger than the ones from Subsection 3.3. We essentially show that, in each case, a countability condition implies the strongest claim of existence of a RIR, an exact and—in the first two cases—singular-exact one. Since Section 7 has shown that already the "weak" claims of existence of an RIR (from theorems 5 and 6; for the "closed" case it is known according to Theorem 1) imply the respective countability conditions, the reasonings of the present section close the circle required for the "comprehensive" claims, from which the single theorems and corollaries follow.

8.2 Closed representations: proof of Corollary 3.

Concerning CRIRs, we have little more to say than Fishburn [8, 9], *viz.* Corollary 3.

According to Subsection 8.1, we instead ought to prove (e.g.)

Corollary 10. *If (F1) and (F2), there is an exact and singular-exact CBRIR of (A, T).*

However, Fishburn already has done most of this work. By proving Corollary 3, we add the little difference concerning our singularity notions as well as a simple equivalence literally not comprised by Corollary 10.

Proof of Corollary 3. Assume ρ is a CBRIR of (A, T), $u : A \to \mathbf{R}, a \mapsto \inf \rho(a)$, and $v : A \to \mathbf{R}, a \mapsto \sup \rho(a)$. It is then obvious from the definitions and from Lemma 2 that ρ is exact iff u is a real IT-homomorphism on A and v is a real TI-homomorphism on A. Thus let us call the FPR (u, v) exact in this case. It is likewise obvious that ρ is singular-exact iff $u(b) = v(a)$ for $a\ I^*\ b$. Call the FPR (u, v) singular-exact in this case. It is now clear that the last two conditions of Corollary 3 are equivalent. As an exact and singular-exact CBRIR *is* a CRIR, it remains to show that, if there is a CRIR, there is an exact and singular-exact FPR. So assume (A, T) has a CRIR.

By Theorem 1 or the reasoning in Section 5, (A, T) has a CBRIR.[87] By Fishburn's [8, Thm. 2] or [9, Thm. 7.6], (A, T) has an *exact* FPR (u', v').[88] Define $u, v : A \to \mathbf{R}$ by $v(a) := v'(a)$ for $a \in A \setminus S_+$, $u(b) := u'(b)$ for $b \in A \setminus S_-$, and $u(b) := v(a) := (u'(b) + v'(a))/2$ for $a\ I^*\ b$.

For $a\ I^*\ b$, Lemma 2 yields $u'(b) \leq v'(a)$. If, moreover, $u'(b) = v'(a)$, then $u(b) = u'(b)$ and $v(a) = v'(a)$. If $u'(b) < v'(a)$ instead, exactness yields the following essential facts: the open interval $(u'(b), v'(a))$ does not intersect with $u'[A] \cup v'[A]$ (the image sets of u', v'), $u'(b) \notin u'[A \setminus [b]_{IT}] \cup v'[A]$, and $v'(a) \notin v'[A \setminus [a]_{TI}] \cup u'[A]$. For these reasons, the relevant inequalities remain true when the 'prime' strokes in u', v' are removed—this means, (u, v) inherits being an exact FPR from (u', v'). By construction, moreover, (u, v) is singular-exact. □

8.3 A representing linear order of "cuts".

In principle, our following proofs just vary Fishburn's [8, 9] proof that (F1) implies existence of an ARIR. However, we use a different approach. While Fishburn applied the Real Homomorphism Theorem to appropriate weak orders on a "doubled" version of A, we apply it to appropriate suborders of one given linear order.[89] The following may be considered a modification of [2].

Call $C \subseteq A$ a *cut* if it is closed under $\precsim_{TI}$, i.e., if $a \precsim_{TI} c \in C$ implies $a \in C$.[90] Let K be the set of these cuts. In order to have a *set* (instead of the proper *class* $\subset$), we use E to denote $\subset \cap (K \times K)$. Now, (K, E) is a linear order.

Proof. Irreflexivity and transitivity of E on K are clear. For trichotomy, assume $C, C' \in K$, $C \neq C'$, and $C \not\subset C'$. Thus, there is some $c \in C \setminus C'$. We show $C' \subset C$.

[87] We need this for the following because of our exegetical doubts according to Fn. 18 above. Also cf. Fishburn's [8] proof of his Thm. 5—for the previous point as well as for our withholding details.

[88] Indeed, by Fishburn's theorem, the FPR may be assumed to be singular-exact in a sense restricted to *Fishburn*'s notion of singularity, but this does not help here.

[89] Fishburn [8, 9] defines one relation on a "doubled" version of A for each of three purposes. In order to apply the Real Homomorphism Theorem, he then has to prove that this relation is a weak order. Using suborders of one linear order, we bypass this step.—However, there may well arise too much clumsy difficulties from our approach which may advise to go along Fishburn's lines instead of ours. We have not yet investigated this suggestion. If other approaches should prove to be much less clumsy, our work will at least contribute to realize this fact.

[90] Our "cuts" are something quite different from the "instants" used in [19] according to the remark ensuing the proof of its Prop. 2—they typically violate its (14).

If $C' = \emptyset$, we are ready. So assume $c' \in C'$. Now, $c' \precsim_{TI} c$ or $c\ TI\ c'$. In the first case, $c' \in C$ by definition of cuts. In the second case, $c \precsim_{TI} c'$; so $c \in C'$ by the definition of cuts—but this contradicts the former assumption of $c \notin C'$. We conclude $c' \in C$, thus $C' \subseteq C$ and, by the assumption of $C \neq C'$, $C' \subset C$.[91] $\square$

By the way, $A \to \mathcal{I}(K, E), a \mapsto \{\, C \in K \mid Ta \subset C \subseteq \precsim_{TI} a \,\}$ is an IR of (A, T) in (K, E).[92] This amounts to another proof that (A, T), if only assumed to consist of an irreflexive $T \subseteq A \times A$ and to satisfy (W), has an IR in some linear order.[93]

8.4 Two special kinds of cuts and their interrelations.

We shall meet cuts of two special kinds:

Lemma 19. *For $a \in A$, Ta and $\precsim_{TI} a$ are cuts.*

Proof. Ta is a cut by (4), and $\precsim_{TI} a$ is a cut by negative transitivity of the weakly ordering TI. $\square$

Furthermore, we shall need some knowledge on their interrelations and on how the latter interrelate with properties of the "source" objects in A:

Lemma 20. *If $Ta \subset \precsim_{TI} b$ and $b \notin S_+$, then $a\ ITITI\ b$.*

Proof. Assume $Ta \subset \precsim_{TI} b$ and $b \notin S_+$. Then, by (4), not $b\ T\ a$. By $b \notin S_+$, neither b nor anything is an IT-maximum of Ib. Hence, if $a\ T\ b$, then there is some c such that $a\ I\ a\ T\ b\ IT\ c\ I\ b$. If $a\ I\ b$, neither a nor anything is in IT-$\mathrm{Max}\, Ib$, so there are $c, c' \in Ib$ such that $a\ IT\ c\ IT\ c'\ I\ b$. $\square$

Lemma 21. *If $a\ (I \cup ITI \cup T)\ b$, then $Ta \subset \precsim_{TI} b$.*

Proof. Assume $a\ I\ b$; so $b \notin Ta$. Additionally assume $c\ T\ a$ for some c. Then $c\ TI\ b$. To sum up, $Ta \subset \precsim_{TI} b$.—Now assume $a\ IT\ c\ I\ b$ for some c. Then, using (3) and the previous, $Ta \subseteq Tc \subset \precsim_{TI} b$. The case of IT follows by reflexivity of I.—Finally, assume $a\ T\ b$ and $c\ T\ a$ for some c as firstly. Then $c\ TT\ b$, so, by transitivity (Lemma 4), $c\ T\ b\ I\ b$ and $c\ TI\ b$—desired as in the first case. $\square$

Lemma 22. *If $a\ T^*\ b$, then $\precsim_{TI} a = Tb$.*

[91] Obviously, the proof works for any binary asymmetric relation on A instead of TI.

[92] This is a modification of [2]. The proof uses Lemma 19 below.

[93] The presumably earliest proof by B. Russell and N. Wiener [21] uses the axiom of choice. Furthermore, it is easy to see that Fishburn's definition [9, Sec. 2.1] of interval orders ((W) replaced by *if $a\ T\ b$ and $a'\ T\ b'$, then $a\ T\ b'$ or $a'\ T\ b$*) is equivalent to that of what Wiener calls a 'relation of complete sequence'. (Thomason [19, 20] adds two relations defined from the first one and calls 'event structures' what results; in principle, however, this is the same.) Fishburn [9, Thm. 2.6] even proves that any interval order (as defined by irreflexivity of T and by (W) or a version of the latter) has a CBIR. Both proofs, and neither ours, do *not* need the axiom of choice.

Proof. Assume $a\ T^*\ b$. By (4), then, $\precsim_{TI}a \subseteq Tb$. For equality assume, additionally, $c\ T\ b$. Then $a\ T^*\ b$ entails *not* $a\ TIT\ b$ and, therefore, *not* $a\ TI\ c$, which means $c \precsim_{TI} a$ (TI weakly ordering A). □

Lemma 23. *If* $\precsim_{TI}a \subset Tb$*, then* $a\ TIT\ b$*.*

Proof. Assume $\precsim_{TI}a \subset Tb$. Hence, $a\ T\ b$. Now, $\precsim_{TI}a$ being *proper* subset of Tb by Lemma 22 rejects the stronger proposition $a\ T^*\ b$. Therefore, $a\ TIT\ b$. □

8.5 Existence of arbitrary real representations.

To complete proofs of our claims on ARIRs—*viz.* Theorem 5 and Corollary 4, we prove the following

Proposition 6. *If* S_- *and* $\{\, a \in A \setminus S_- \mid N_{IT}a = \emptyset \,\}$ *are* IT*-separable and, furthermore,* $A \setminus S_+$ *is* TI*-separable,* (A, T) *has an exact and singular-exact ABRIR.*

Note that here is a condition logically (or set-theoretically) weaker than our original condition of singular- and regular-separability of (A, T) (even weaker than the respective condition yielded by Theorem 3). The new condition is entailed by the original one and, therefore, necessary for real representability, as well. Indeed, by Lemma 16 a subset of $A \setminus S_-$ is IT-separable if $A \setminus S_-$ is.

Idea of proof. Fishburn would start a proof of Proposition 6 by "merging" some of the equivalence classes associated with the weak order on a doubled version of A which he has defined in his proof of his part of our Theorem 1. *We* are replacing such "merging" by assigning same cuts in some cases of "overlapping" or of precedence with respect to IT or to TI. "Translated" into Fishburn's method, we are (in general) merging much more equivalence classes than Fishburn ever did (in [8, 9]).[94]

Proof of Proposition 6. First part of construction. For brevity, let $A_1 := S_-$, $A_2 := \{\, a \in A \setminus S_- \mid N_{IT}a = \emptyset \,\}$ $A_3 := A \setminus S_+$; furthermore $K_1 := \{\, Ta \mid a \in A_1 \,\}$, $K_2 := \{\, Ta \mid a \in A_2 \,\}$, and $K_3 := \{\, \precsim_{TI}a \mid a \in A_3 \,\}$. For $a \in A$, let

$$\gamma_-(a) := \begin{cases} Ta & \text{if } a \in S_- \cup A_2; \\ Tb & \text{if } a \notin S_- \text{ and } b\ N_{IT}\ a. \end{cases}$$

$\gamma_-(a)$ is well-defined in the second case by (3) and lemmas 9f. (strong congruency of N_{IT}). By Lemma 19, $\gamma_-(a)$ is an element of K in both cases. Thus we have defined a map γ_- from A into K. Since—by Lemma 14—$b\ N_{IT}\ a$ implies $b \in S_-$, the image set $\gamma_-[A]$ is $K_1 \cup K_2$.

[94] *Fishburn* did not merge more than *two* equivalence classes, while *we* (in Fishburn's terms) are merging *three* of them in some cases. E.g., we are merging respective equivalence classes of a^+, b^-, and c^- (one of Fishburn's notations) whenever $a \in A \setminus S_+$, $c \in A \setminus S_-$, and $a\ T^*\ b\ N_{IT}\ c$.

Next let, for $a \in A$,

$$\gamma_+(a) := \begin{cases} Tb & \text{if } a\ I^*\ b; \\ \precsim_{TI} a & \text{if } a \notin S_+. \end{cases}$$

By (3) and strong congruency of I^* (Lemma 10), $\gamma_+(a)$ is well-defined in the first case. $a\ I^*\ b$ implies $a \in S_+$ (Proposition 1), so the two cases exclude each other and exhaust all possibilities, indeed. As $\precsim_{TI} a$ is a cut (Lemma 19) and by the above reasoning concerning γ_-, $\gamma_+(a)$ is in K again in both cases, so we have a map $\gamma_+ : A \to K$. Since $a\ I^*\ b$ also implies $b \in S_-$, its image set $\gamma_+[A]$ is $K_1 \cup K_3$. Thus, $\gamma_-[A] \cup \gamma_+[A] = K_1 \cup K_2 \cup K_3$. Call the latter set K_0. According to the above thoughts about γ_- and γ_+, it is a subset of K. Recall that, now, (K_0, E) is a linear order and, therefore, a weak order.

Separability. Assume the hypothesis of Proposition 6; in particular, there are countable D_1 being IT-dense in S_-, D_2 IT-dense in A_2, and D_3 TI-dense in $A \setminus S_+$ (so $D_k \subseteq A_k$ for $k = 1, 2, 3$). In order to apply the Real Homomorphism Theorem, we show that K_0 is E-separable. Let $D'_1 := \{\, Td \mid d \in D_1 \,\}$, $D'_2 := \{\, Td \mid d \in D_2 \,\}$, and $D'_3 := \{\, \precsim_{TI} d \mid d \in D_3 \,\}$. We show that $D' := D'_1 \cup D'_2 \cup D'_3$—being countable—moreover is E-dense in K_0.

Clearly, $D'_i \subseteq K_i$ for $i = 1, 2, 3$; so $D' \subseteq K_0$. For density, assume $C, C' \in K_0$ and $C \subset C'$. We distinguish 9 cases numbered $i.j$, $i, j = 1, 2, 3$, where Case $i.j$ means assuming $C \in K_i$ and $C' \in K_j$. In Case $i.j$ we may pick a from A_i and b from A_j such that

$$C\,[C'] = \begin{cases} Ta\,[Tb] & \text{if } i\,[j] < 3; \\ \precsim_{TI} a\,[\precsim_{TI} b] & \text{if } i\,[j] = 3. \end{cases}$$

We shall write $[c_1, c_2]_k$ for $\{\, c_0 \in D_k \mid c_1 \precsim_{R_k} c_0 \precsim_{R_k} c_2 \,\}$ where $R_1 := R_2 := IT$ and $R_3 := TI$. (Hence, 1 indicates (S_-, IT), 2 indicates (A_2, IT), and 3 indicates (S_+, TI) being "reference order" for taking "intervals" [notion generalized from linear orders].—c_1 or c_2 may be no member of $[c_1, c_2]_k$.) Furthermore, we shall write K^* for $\{\, C_0 \in K_0 \mid C \subseteq C_0 \subseteq C' \,\} \cap D'$ (recall $\subseteq = \precsim_E$ on K). Thus, we are trying to show $K^* \neq \emptyset$.

Keep in mind that $i, j < 3$ implies $Ta \subset Tb$, so $a\ IT\ b$ follows by (3).

Case 1.1. Here, by hypothesis and the previous remark, we may pick d from $[a, b]_1$. Then $Td \in K^*$ by (3).

Case 1.2. Again, $a\ IT\ b$. By $j = 2$, $N_{IT} b = \emptyset$, so we may pick c from $a(ITIT) \cap (IT)b$. If there is some $c' \in N_{IT} c$, then $a\ IT\ c'\ IT\ b$, and there is (by Lemma 14) some $d \in [a, c']_1$ such that $Td \in K^*$ by applying (3) and transitivity of IT.—Otherwise, $N_{IT} c = \emptyset$, so there is some $d \in [a, c]_1 \cup [c, b]_2$, and $Td \in K^*$ essentially as before.

Case 2.1. Here, $a\ ITIT\ c\ IT\ b$ for some c by $a \notin S_-$ and Lemma 6. Continue almost as in Case 1.2 (using $[c', b]_1$, $[a, c]_2$, and $[c, b]_1$ instead).

Case 2.2. By hypothesis, pick d from $[a, b]_2$ and proceed analogously to Case 1.1.

For the following cases, recall that $\precsim_{TI}$ is involved when $i = 3$ or $j = 3$.

Case 3.3. Here, $\precsim_{TI} a \subset \precsim_{TI} b$ and, therefore (Lemma 19 and $b \notin \precsim_{TI} a$), $a\ TI\ b$. Now $\precsim_{TI} d \in K^*$ for some $d \in [a, b]_3$.

Case 1.3. By Lemma 20, $a(ITIT) \cap Ib \neq \emptyset$. If there even is some $c \in a(ITIT) \cap Ib \cap S_-$, there is some $d \in [a, c]_1$ such that $Td \in K^*$ by (3) and Lemma 21. Otherwise, there might be c, c' such that $c'\ N_{IT}\ c\ I\ b$. In this subcase, $c' \in a(IT) \cap N_{IT} c \cap S_-$ as in Case 1.2, so pick d from $[a, c']_1$ and continue as above (using the ITI case of Lemma 21). If the two previous subcases apply to *no* c, there only may be c in $a(IT) \cap Ib \cap A_2$. By $b \notin S_+$, then, there is some $c' \in c(IT) \cap Ib$, which (being element of $a(ITIT) \cap Ib$) must be in A_2, as well. Now, there is some $d \in [c, c']_2$ such that $Td \in K^*$ by (3) and Lemma 21.

Case 2.3. If there are $c, c' \in S_-$ such that $a\ IT\ c\ IT\ c'\ ITI\ b$, one may pick d from $[c, c']_1$ in order to get, by (3) and Lemma 21, $Td \in K^*$. Otherwise, by $a \notin S_-$ and lemmas 6 and 20, there are c_1, c_2, c_3 such that $a\ ITIT\ c_1\ IT\ c_2\ IT\ c_3\ ITI\ b$. In this subcase, two out of c_1, c_2, c_3 cannot both be in S_-. Among these two, only one c can meet $N_{IT} c \neq \emptyset$ (Lemma 14; first subcase of 2.3 being excluded here). Concerning the other—c', say—, $c' \in A_2$, and for some $d \in [a, c']_2$, $Td \in K^*$ as earlier.

Cases 3.1 and 3.2. By Lemma 23, $a(TI) \cap Tb \neq \emptyset$. If there even is some $c \in (a(TI) \cap Tb) \setminus S_+$, there is some $d \in [a, c]_3$ yielding $\precsim_{TI} d \in K^*$ by lemmas 1, 3, and 21 (case of T), and by (W). Otherwise, $a(TI) \cap Tb \subseteq S_+$. If, now, there are $c, c' \in S_+$ such that $a\ TI\ c\ TI\ c'\ T\ b$, then there is $d \in [c, c']_3$, and $\precsim_{TI} d \in K^*$ similarly as before. Otherwise again, $a(TI) \cap Tb \subseteq a(T^*I)$. Then, by Lemma 22, $\precsim_{TI} a = Tc$ for some $c \in aT^*$, so one of the above cases with $i, j < 3$ applies.

Second part of construction. Now that K_0 is E-separable, we may choose a real E-homomorphism φ on K_0. Thereby, we may define, for $a \in A$,

$$\begin{aligned}
\rho_-(a) &:= \begin{cases} \geq \varphi(\gamma_-(a)) & \text{if } a \in S_-; \\ > \varphi(\gamma_-(a)) & \text{if } a \notin S_-; \end{cases} \\
\rho_+(a) &:= \begin{cases} \leq \varphi(\gamma_+(a)) & \text{if } a \in S_+; \\ < \varphi(\gamma_+(a)) & \text{if } a \notin S_+; \end{cases} \\
\rho(a) &:= \rho_-(a) \cap \rho_+(a).
\end{aligned}$$

(Now, e.g., $\rho(a) = [\varphi(\gamma_-(a)), \varphi(\gamma_+(a))]$ if $a \in S_- \cap S_+$.)

It remains to show that this ρ is an exact and singular-exact ABRIR.

A lemma. Proposition 1 yields

$$\text{(6)} \qquad \textit{If} \quad a\ I^*\ b, \quad \textit{then} \quad \gamma_+(a) = Tb = \gamma_-(b).$$

Another lemma. Suppose $a \in I^*b \setminus S$. Then by Lemma 5, *not* $a\ I^*\ a$, i.e., *not* $a \in IT\text{-Max}\, Ia$. Hence, by Lemma 8, *not* $a \sim_{IT} b$. On the other hand, $b \in IT\text{-Max}\, Ia$, hence $a \precsim_{IT} b$. To conclude:

$$\text{(7)} \qquad \textit{If} \quad a \in I^*b \setminus S, \quad \textit{then} \quad a\ IT\ b.$$

Image set consisting of bounded real intervals. For $\rho(a) \in \mathcal{I}(\mathbf{R}, <)$ $(a \in A)$, it clearly suffices to show that $\rho(a) \neq \emptyset$. It will then be clear that $\rho(a)$ has $\varphi(\gamma_-(a))$ as a lower and $\varphi(\gamma_+(a))$ as an upper bound.—First, consider $a \in S$. By Lemma 5, then, $S \subseteq S_- \cup S_+$, so, by (6), $\gamma_-(a) = Ta = \gamma_+(a)$ and $\rho(a) = \{\varphi(Ta)\} \neq \emptyset$.—Now suppose $a \notin S$. We are going to show $\gamma_-(a) \subset \gamma_+(a)$ in order to obtain $\inf \rho(a) = \varphi(\gamma_-(a)) < \varphi(\gamma_+(a)) = \sup \rho(a)$ and, thus, $(\inf \rho(a) + \sup \rho(a))/2 \in \rho(a) \neq \emptyset$. There are four cases according to which two of S_-, $A \setminus S_-$, $A \setminus S_+$, S_+ a belongs. Recall that, by trichotomy of E, $C \subset C'$ iff $C' \setminus C \neq \emptyset$ whenever $C, C' \in K$.

$a \in S_+ \cup S_-$: Here, there is some $b \in aI^*$, so there is some $c \in aI \cap Tb$ by (7). The latter entails $c \in Tb \setminus Ta$, hence $\gamma_-(a) = Ta \subset Tb = \gamma_+(a)$.

$a \in S_+ \setminus S_-$: If $N_{IT}a = \emptyset$, then $\gamma_-(a) = Ta \subset Tb = \gamma_+(a)$ nearly as before.[95] Otherwise, there are b, c such that $\gamma_-(a) = Tb$, $\gamma_+(a) = Tc$, and $b\ N_{IT}\ a\ I^*\ c$. (7) yields $b\ IT\ a\ IT\ c$ and, by transitivity of the weakly ordering IT, $b\ IT\ c$. Now, $\gamma_-(a) = Tb \subset Tc = \gamma_+(a)$ nearly as in the previous case.

$a \in S_- \setminus S_+$: Here, $\gamma_-(a) = Ta \subset {\precsim_{TI}}a = \gamma_+(a)$ by $a \in {\precsim_{TI}}a \setminus Ta$.

$a \notin S_- \cup S_+$: If $N_{IT}a = \emptyset$, the previous proof applies, again. Otherwise, there is some $b \in N_{IT}a$, so $a \in {\precsim_{TI}}a \setminus Tb$ by (5), and, finally, $\gamma_-(a) = Tb \subset {\precsim_{TI}}a = \gamma_+(a)$.

ρ interval representation. We have to show that $a\ T\ b$ iff

$$\rho_+(a) \cap \rho_-(b) = \emptyset. \tag{8}$$

There are four cases according to which two of S_-, $A \setminus S_-$ a belongs and to which two of S_+, $A \setminus S_+$ b belongs.

$a \in S_+$, $b \in S_-$: Here, there is some $c \in I^*a$ such that $\rho_+(a) = {\leq}\varphi(Tc)$. Furthermore $\rho_-(b) = {\geq}\varphi(Tb)$. We need a number of equivalences dependant on these conditions. The first is *$a\ T\ b$ iff $c\ IT\ b$.* $a\ T\ b$ implies $c\ I\ a\ T\ b$—one direction. $c \in I^*a$ means $c \in IT\text{-Max}\,Ia$, so $c\ IT\ b$ entails *not* $a\ I\ b$. Similarly, $c\ I^*\ a$ implies $a \precsim_{IT} b$, so $c\ IT\ b$ entails $a\ IT\ b$ (Lemma 1) and *not* $b\ T\ a$ (cf. (5))—the equivalence is established.—Detailed proofs of the further equivalences may be left to the reader: ($a\ T\ b$ iff) $c\ IT\ b$ iff (cf. (3)) $Tc \subset Tb$ iff $\sup \rho_+(a) < \inf \rho_-(b)$ iff (8).

$a \in S_+$, $b \notin S_-$: Consider $\rho_+(a)$ and c as before. If $N_{IT}b = \emptyset$, then $\rho_-(b) = {>}\varphi(Tb)$. Now, $a\ T\ b$ entails (8) as before, again. For the other direction, (8) at first only yields $\sup \rho_+(a) \leq \inf \rho_-(b)$, $Tc \subseteq Tb$, and $c \precsim_{IT} b$. However, $c \in S_-$ while $b \notin S_-$. Thus, as S_- is IT-congruent, *not* $c \sim_{IT} b$, so $c\ IT\ b$ obtains, and $a\ T\ b$ is derived as in the previous case.—If, on the other hand, there is $d \in N_{IT}b$, we get $\rho_-(b) = {>}\varphi(Td)$. Some appropriate modification of the above yields $a\ T\ b$ iff $c \precsim_{IT} d$ iff $Tc \subseteq Td$ iff $\sup \rho_+(a) \leq \inf \rho_-(b)$ iff (8).

[95] Distinction of cases according to the cases of the definitions of γ_-, γ_+ here and below might be shorter; however, we suppose that it rather would be less perspicuous.

$a \notin S_+$, $b \in S_-$: Here, $\rho_+(a) = {<}\varphi(\precsim_{TI} a)$ and $\rho_-(b) = {\geq}\varphi(Tb)$. Now, $a\ T\ b$ iff $\precsim_{TI} a \subseteq Tb$ iff $\sup \rho_+(a) \leq \inf \rho_-(b)$ iff (8).

$a \notin S_+$, $b \notin S_-$: Here, $\rho_+(a) = {<}\varphi(\precsim_{TI} a)$ as previously. If $N_{IT} b = \emptyset$, then $\rho_-(b)$ as previously, and the previous reasoning applies, again. Otherwise, pick d from $N_{IT} b$. Then, $\rho_-(b) = {>}\varphi(Td)$ as in the second case above, and $a\ T\ b$ follows from $\precsim_{TI} a \subseteq Td$ by (W). $a\ T\ b$, conversely, does not allow $d\ T\ a$, since the latter would contradict $d\ N_{IT}\ b$ by (1). Neither, it allows $d\ I\ a$, since this would contradict $d\ N_{IT}\ b$ by $a \notin S_+$ and (W) (consider $d\ IT\ d'\ IT\ d''$, $d', d'' \in Ia$, then $d\ IT\ d'\ IT\ b$). To conclude, $a\ T\ b$ yields $a\ T\ d$; so $a\ T\ b$ iff $\precsim_{TI} a \subseteq Td$ iff $\sup \rho_+(a) \leq \inf \rho_-(b)$ iff (8).

Exactness. By IT-congruency of S_- and of $A \setminus S_-$ (Lemma 11; complements are congruent because the whole set is), strong IT-congruency of N_{IT}, and (3), $\rho_-(a) = \rho_-(b)$ if $a \sim_{IT} b$. By TI-congruency of S_+ and of $A \setminus S_+$, strong TI-congruency of I^*, and (4), $\rho_+(a) = \rho_+(b)$ if $a \sim_{TI} b$.

Singular-exactness. By (6), $\rho(a) \cap \rho(b) = \{\varphi(Tb)\}$ if $a\ I^*\ b$. □

8.6 Existence of open real representations.

To complete proofs of our claims on ORIRs—*viz.* Theorem 6 and corollaries 2 and 5, we prove the following

Proposition 7. *If (A, T) is singular-countable and regular-separable, it has a WFPR (u, v) such that u is a real IT-homomorphism and v is a real TI-homomorphism.*

Proof. Assume (A, T) is singular-countable and regular-separable. This is a special case of the hypotheses in Proposition 6, so there are $\varphi, \rho_-, \rho_+, \rho$ as in our proof of Proposition 6.

By $\mathbf{R}^+$ we denote the set of real numbers $r > 0$. Let $\exp : \mathbf{R} \to \mathbf{R}^+, r \mapsto e^r$.[96] Then $\exp \circ \varphi$ is a real E-homomorphism like φ and could have been used for defining ρ, as well. Without loss of generality, therefore, we may assume that each $\varphi(C) > 0$ ($C \in K_0$); so each $\rho(a) \in \mathcal{I}(\mathbf{R}^+, <)$ ($a \in A$). Let $Q_0 := \{\, \inf \rho(a) \mid a \in S_- \,\} \subseteq \mathbf{R}^+$. Since ρ is exact, Q_0 is countable by singular-countability. Thus, we may choose some sequence $(q_n)_{n \in \mathbf{N}}$ such that $n \mapsto q_n$ maps $\mathbf{N}$ one-to-one *onto* Q_0. (We are somewhat replacing each point q_n of Q_0 by an interval of length 2^{-n}.)

Now let $\psi_-, \psi_+ : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\psi_-(r) := r + \sum_{n \in \mathbf{N}, q_n < r} 2^{-n}$ and $\psi_+(r) := r + \sum_{n \in \mathbf{N}, q_n \leq r} 2^{-n}$. (The sums are subseries of a convergent geometric series and have a value independent on the order by which partial sums are taken.)[97]

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[96] e is Euler's number 2,71828…

[97] More precisely, let $p_n^{(r)} := 2^{-n}$ if $q_n < r$ and $p_n^{(r)} := 0$ otherwise; and let $q_n^{(r)} := 2^{-n}$ if $q_n \leq r$ and $q_n^{(r)} := 0$ otherwise. Then $\psi_-(r) := r + \sum_{n=0}^{\infty} p_n^{(r)}$ and $\psi_+(r) := r + \sum_{n=0}^{\infty} q_n^{(r)}$.

Next let, for $a \in A$,

$$u(a) := \begin{cases} \psi_-(\inf \rho(a)) & \text{if } a \in S_-; \\ \psi_+(\inf \rho(a)) & \text{if } a \in A \setminus S_-; \end{cases}$$
$$v(a) := \begin{cases} \psi_+(\sup \rho(a)) & \text{if } a \in S_+; \\ \psi_-(\sup \rho(a)) & \text{if } a \in A \setminus S_+; \end{cases}$$

This defines $u, v : A \to \mathbf{R}^+$. (u, v) straightforwardly proves to be a WFPR of (A, T) having the additional 'homorphism' property required by the proposition.

However, one might overlook simple proofs, so we try to present one as follows.

We consider $a, b \in A$ and show that $a\ T\ b$ iff

$$v(a) \leq u(b). \tag{9}$$

But we know that $a\ T\ b$ is equivalent to (8), hence it suffices to show that (9) is equivalent to (8).

Assume (8). Then $\sup \rho(a) \leq \inf \rho(b)$. If $\sup \rho(a) < \inf \rho(b)$, then (9) follows from

$$\psi_-(r) \leq \psi_+(r) < \psi_-(r') \leq \psi_+(r') \qquad \text{for} \quad 0 < r < r'. \tag{10}$$

If $\sup \rho(a) = \inf \rho(b) =: r_0$, then one of $\rho_+(a), \rho_-(b)$ is open; hence $a \in A \setminus S_+$ or $b \in A \setminus S_-$ by definitions of ρ_-, ρ_+, furthermore $v(a) = \psi_-(r_0)$ or $u(b) = \psi_+(r_0)$, and this yields (9).

Now assume (8) does *not* hold. In this case, $\inf \rho(b) \leq \sup \rho(a)$. If $\inf \rho(b) < \sup \rho(a)$, then (10) yields $u(b) < v(a)$. If $\inf \rho(b) = \sup \rho(a) = r_0$, then $\rho(a) \cap \rho(b) = \{r_0\}$.[98] Lemma 2 leads from this to $a\ I^*\ b$; hence Proposition 1 yields $r_0 \in Q_0$ and $u(b) = \psi_-(r_0) < \psi_+(r_0) = v(a)$. Thus, *not* (8) implies *not* (9), and we know that (u, v) is a WFPR of (A, T).

That u is a *weak*[99] real IT-homomorphism and v a weak real TI-homomorphism follows from Lemma 2 and ρ being an IR. That they are even *strong* ones follows from exactness of ρ and from respective congruency of $S_-, A \setminus S_-, A_+, A \setminus S_+$. □

9 Proofs concerning semiorders.

This section deals with Subsection 3.7.

Corollaries 6 through 8, however, are straightforward from the definitions and Lemma 2—recognizing that (A, T) is a semiorder iff $a\ IT\ b\ TI\ a$ for *no* $a, b \in A$.

Proof of Theorem 9. Assume (A, T) is the natural interval order of all real intervals having length 1 and containing their lower bounds and, for *reductio*, ρ is a strictly semiorderlike RIR of (A, T). Clearly, then, (A, T) is a semiorder. If $r \in \mathbf{R}$, let $a_r := [r, r+1)$ and $b_r := [r, r+1]$. Some applications of Lemma 2 are ensuing: For $r \in \mathbf{R}$, there is some $q_r \in \rho(b_{r-1}) \cap \rho(b_r)$.

[98] Assumption of *not* (8) is still at work.

[99] Cf. Section 7.

Moreover, $a_r\ T\ b_{r+1}\ I\ b_r$, so (by strict semiorderlikeness) $\rho(a_r)$ $<$-exceeds $\rho(b_r)$ below, i.e., $p_r < q$ for some $p_r \in \rho(a_r)$ and all $q \in \rho(b_r)$. In particular, $p_r < q_r$. If $r < r'$, then $b_{r-1}\ T\ a_{r'}$ and, hence, $q_r < p_{r'}$. Therefore, the uncountably many open real intervals in $\{\,(p_r, q_r) \mid r \in \mathbf{R}\,\}$ are pairwise disjoint, and each intersects with the countable set of rational numbers—a contradiction. □

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